Aspects of Pseudolocality in Ricci Flow

by

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Contents

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Declarations

Chapter 2 contains numerous classical results that will later be required. Chapter 3 provides the details behind a compactness theorem already appearing within the literature, most notably within the paper [ST17] of Miles Simon and Peter Topping. Chapter 4 is taken from the paper [MT18], which is joint work with my supervisor Peter Topping. Chapter 5 is taken from the paper [McL18], which is work of the author undertaken during his doctoral studies. Extensive references are provided within the text when the results are not my own work.

I declare that to the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise indicated. This thesis has not been submitted for a degree at any other university

Abstract

This thesis explores two separate problems related to the phenomenon of *pseudolocality* within Ricci flow.

First, we consider the regularity of noncollapsed three-dimensional Ricci limit spaces via Ricci flow. We introduce a new weakened notion of Ricci flow, termed *Pyramid Ricci flow*, and use it to establish that a noncollapsed three-dimensional Ricci limit space is homeomorphic to a smooth manifold via a globally-defined homeomorphism that is bi-Hölder once restricted to any compact subset. We include the full details of a well-known compactness result which this work relies upon.

Second, we consider the *pseudolocality* phenomenon in an *almost-hyperbolic* setting. We obtain an improved pseudolocality result for Ricci flows on two-dimensional surfaces that are initially *almost-hyperbolic* on large hyperbolic balls. We prove that, at the central point of the hyperbolic ball, the Gauss curvature remains close to the hyperbolic value for a time that grows exponentially in the radius of the ball. This two-dimensional result allows us to precisely conjecture how the phenomenon should appear in the higher dimensional setting.

Chapter 1

Introduction

Beginning with the pioneering introduction of the *harmonic map heat flow* by Eells and Sampson in 1964 (see [ES64]), *geometric* flows have been extensively utilised to tackle both geometric and topological problems. First introduced by Hamilton in his ground-breaking article [Ham82] in 1982, *Ricci flow* is the following partial differential equation (PDE)

$$\frac{\partial g}{\partial t}(t) = -2\mathrm{Ric}_{g(t)},$$

posed on the space of positive-definite symmetric two-tensor fields on a smooth manifold $\mathcal{M}$, where $\mathrm{Ric}_{g(t)}$ denotes the Ricci curvature tensor of the solution $(g(t))_{t\in[0,T]}$. Heuristically, this is an analogue of the standard heat equation, and one hopes to evolve the metric tensor via Ricci flow to improve its regularity, with the aim of being able to deduce both geometric and topological conclusions from the existence of metrics $g(t)$, for $t > 0$, enjoying better regularity.

Hamilton was primarily interested in attempting to use the Ricci flow to prove the *Thurston Geometrisation Conjecture* (see [Thu82]) which classifies three-manifolds and contains the famous Poincaré conjecture as a special case. Hamilton laid the foundations upon which Perelman built to fully prove the *Thurston Geometrisation Conjecture* in his ground-breaking papers [Per02], [Per03-I] and [Per03-II]. Ricci flow has been central to several other remarkable achievements, including the $\frac{1}{4}$-pinched differentiable sphere theorem of Brendle and Schoen in [BS09], and the recent full resolution of the three-dimensional conjecture of Anderson-Cheeger-Colding-Tian (often called the ACCT conjecture) obtained by Simon and Topping in [ST17].

The three-dimensional ACCT conjecture asserts that weakly noncollapsed three-dimensional Ricci limit spaces (Gromov-Hausdorff limits of sequences of manifolds satisfying global lower Ricci curvature bounds and having the volume of a *single* unit ball controlled from below, see Section 2.9) are homeomorphic to manifolds. In earlier work, [Sim12], Simon establishes that the conclusion of the three-dimensional ACCT conjecture is valid under the stronger globally noncollapsed assumption that the volume of *every* unit ball is uniformly controlled from below. This is achieved by using Ricci flow to globally mollify such a metric; that is, to obtain a flow that starts from such a metric, enjoys C/t curvature decay for positive times $t > 0$ and, crucially, propagates the initial Ricci lower bound forward in time in the sense that the flow enjoys some *time-independent* Ricci lower bound for a definite amount of time.

In their works [ST16] and [ST17], Simon and Topping localise the global mollification techniques from [Sim12]. In particular, they refine and extend an approach of Hochard in [Hoc16] to establish that, on compactly contained local regions of a smooth three-manifold with a global Ricci lower bound and the volume of a *single* unit ball uniformly controlled from below, a notion of local Ricci flow on the given region can be run that initially agrees with the given metric on the local region, enjoys C/t curvature decay for positive times $t > 0$ and, locally propagates the initial Ricci lower bound forward in time in the sense that throughout the given region the flow enjoys a *time-independent* Ricci lower bound for a definite amount of time. By using these local Ricci flows to locally mollify each element of the sequence of three-manifolds converging to the weakly noncollapsed three-dimensional Ricci limit space, Simon and Topping are able to establish the three-dimensional ACCT conjecture in full generality.

Within this thesis we examine two main themes within the theory of Ricci flow. The first, which is joint work with Peter Topping, is the introduction of a new notion of local Ricci flow in dimension three called *'Pyramid Ricci flow'*. These *'Pyramid Ricci flows'* are defined on subsets of spacetime which are not parabolic cylinders. This weakened approach allows us to start *'Pyramid Ricci flows'* in situations ill-suited to admitting classical Ricci flow solutions. Being precise, given a smooth, complete Riemannian three-manifold $(\mathcal{M}, g_0, x_0)$ satisfying, for given $v_0, \alpha_0 > 0$, the Ricci lower bound $\mathrm{Ric}_{g_0} \geq -\alpha_0$ throughout $\mathcal{M}$ and the weakly noncollapsed condition that $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0$, then for any $k \in \mathbb{N}$, we prove the existence of a smooth Ricci flow $g_k(t)$ that is defined on a subset of spacetime that contains, for each $m \in \{1, \ldots, k\}$, the cylinder $\mathbb{B}_{g_0}(x_0, m) \times [0, T_m]$, where crucially $T_m > 0$ depends only on α_0, v_0 and m, and in particular *not* on k. Further, the flow enjoys local curvature bounds on the set $\mathbb{B}_{g_0}(x_0, m) \times (0, T_m]$, which again depend only on α_0, v_0 and m.

As the distance from the central point x_0 increases, not only does the existence time of the flow decrease, but the C/t curvature decay estimate that we obtain on the flow $g(t)$ worsens. This is in contrast to the *partial* Ricci flow construction of Hochard, see [Hoc16], and is essential to obtain the uniform estimates on the domain of existence. Another distinction to partial Ricci flows is that by virtue of the theory of Miles Simon and Peter Topping in [ST16, ST17], in particular the so-called Double Bootstrap lemma, our flows have lower Ricci bounds that do not degenerate as $t \downarrow 0$. These uniform lower Ricci bounds will be crucial for obtaining our bi-Hölder estimates in

Theorem 1.0.1 below.

Our considerations of *'Pyramid Ricci flow'* culminates in the following result asserting that noncollapsed three-dimensional Ricci limit spaces are globally smooth manifolds.

Theorem 1.0.1 (***Ricci limit spaces are globally smooth manifolds; Theorem 1.1 in [MT18]***). *Suppose that $\left(\mathcal{M}_i^3, g_i, x_i\right)$ is a sequence of complete, smooth, pointed Riemannian three-manifolds such that for some $\alpha_0 > 0$ and $v_0 > 0$, and for all $i \in \mathbb{N}$, we have $\mathrm{Ric}_{g_i} \geq -\alpha_0$ throughout $\mathcal{M}_i$, and $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$.*

Then there exist a smooth manifold M, a point $x_0 \in M$, and a complete distance metric $d : M \times M \to [0, \infty)$ generating the same topology as M such that after passing to a subsequence in i we have

$$\left(\mathcal{M}_i^3, d_{g_i}, x_i\right) \to (M, d, x_0),$$

in the pointed Gromov-Hausdorff sense, and if g is any smooth complete Riemannian metric on M then the identity map $(M, d) \to (M, d_g)$ is locally bi-Hölder.

Secondly, we consider the so-called pseudolocality phenomenon on large hyperbolic balls. Roughly speaking, Perelman's pseudolocality theorem (originally appearing in Section 10 of [Per02]) asserts that if a region is initially well-controlled (in some sense) then it cannot suddenly look highly non-trivial. Hence there is a definite delay before regions of large curvature can significantly affect regions of controlled curvature. Control of this form is simply not true for solutions to the standard linear heat equation, and illustrates that the inherent nonlinearity in the Ricci flow equation gives rise to an advantageous damping affect not present in the linear setting.

Suppose $g(t)$ is a complete smooth Ricci flow of bounded curvature (see Section 2.3 for definitions) on a smooth n-manifold $\mathcal{M}^n$, such that, for some $r > 0$, we have that $(\mathbb{B}_{g(0)}(x_0, r), g(0))$ is isometric to a Euclidean ball of radius r, where $x_0 \in \mathcal{M}$. Then Theorem 10.3 in [Per02] (Theorem 2.6.2 here) yields that at the point x_0 the curvature of $g(t)$ remains bounded (in the pointwise sense) for a time that is quadratic in the radius r.

In the hyperbolic setting, namely if we assume that initially $(\mathbb{B}_{g(0)}(x_0, r), g(0))$ is isometric to a hyperbolic disc of radius r, we may again appeal to Theorem 2.6.2. However, the requirement that $|\mathrm{Rm}|_{g(0)} \leq r^{-2}$ throughout $\mathbb{B}_{g(0)}(0, r)$ limits us to considering only radii $r \in (0, 1]$, so the curvature at the point x_0 can only be controlled for an order 1 time, irrespective of how large r is.

The second main achievement of this thesis is to establish that, in dimension two, if a sufficiently large initial disc is isometric to a hyperbolic disc of the same radius, the Gauss curvature at the central point remains bounded for a time that is exponential in the radius.

Theorem 1.0.2 (**Improved control time with equality on large initial ball; Theorem 1.2 in [McL18]**)**.** *For any $\alpha \in (0, 1]$ there exist constants $\mathcal{R} = \mathcal{R}(\alpha) > 0$ and $c = c(\alpha) > 0$ for which* *the following holds:*

Let $R \geq \mathcal{R}$ and assume that $g(t)$ is a complete smooth Ricci flow on a smooth surface $\mathcal{M}$, defined for all $t \in [0,T]$ for some $T > 0$, and such that, for some $x \in \mathcal{M}$, we have that $\left(\mathbb{B}_{g(0)}(x,R), g(0)\right)$ is isometric to a hyperbolic disc of radius R. Then at the point x we have

$$-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}}(x) \leq -1+\alpha \qquad \textit{for all} \qquad 0 \leq t \leq \mathcal{T}_{max} := \min\left\{T, e^{cR}\right\}.$$

Further, in Chapter 5, we are able to weaken the initial hypothesis to being *almost-hyperbolic*, see Theorem 5.1.6 for precise details. This two-dimensional result allows us to precisely conjecture how the phenomenon should appear in higher dimensions.

Conjecture 1 (Improved time control with equality on initial ball; Conjecture 1 in [McL18]). *Let $n \in \mathbb{N}$ such that $n \geq 3$. There are constants $\mathcal{A} = \mathcal{A}(n) > 0$, $c = c(n) > 0$ and $\mathcal{R} = \mathcal{R}(n) > 0$ for which the following holds:*

Let $R \geq \mathcal{R}$ and suppose that $g(t)$ is a smooth complete Ricci flow of bounded curvature on a smooth n-dimensional manifold $\mathcal{M}$, defined for all $t \in [0,T]$ for some $T > 0$, and, for some $x \in \mathcal{M}$, suppose we have that $\left(\mathbb{B}_{g(0)}(x,R), g(0)\right)$ is isometric to a hyperbolic ball of radius R. Then at $x \in \mathcal{M}$ we have that

$$|\mathrm{Rm}|_{g(t)}(x) \leq \mathcal{A} \qquad \textit{for all} \qquad 0 \leq t \leq \mathcal{T}_{max} := \min\{T, e^{cR}\}.$$

We further expect that the hypotheses of the previous conjecture can be weakened to *almost-hyperbolic* hypotheses in a similar spirit to the hypotheses of Theorem 5.1.6.

The thesis is structured as follows. In Chapter 2 we fix our notation and provide a summary of background material we require later. Most of the material, with the exception of Section 2.4, is classical and may be skipped by the experienced reader. In Section 2.4 we provide a swift summary of the recent local Ricci flow results of Miles Simon and Peter Topping in [ST16, ST17], upon which our *'Pyramid Ricci flow'* construction in Chapter 4 relies.

In Chapter 3 we provide a detailed proof of a localised Hamilton-Cheeger-Gromov compactness theorem, Theorem 3.2.1, in the incomplete setting. That this is possible, and that the proof carries across more or less verbatim, is well-known and we include the details for completeness.

In Chapter 4 we introduce and use our weakened notion of *'Pyramid Ricci flow'* to prove that noncollapsed three dimensional Ricci limit spaces are globally smooth manifolds. Within this chapter we rely upon the *'Pyramid Ricci flow compactness theorem'*, Theorem 4.5.1. This itself relies upon the local version of the Cheeger-Gromov-Hamilton compactness theorem for Ricci flows, Theorem 3.6.1, which is already implicit in [ST17], and is proven in Chapter 3. Theorem 3.6.1 is itself reliant on the localised Hamilton-Cheeger-Gromov compactness theorem, Theorem

3.2.1.

Finally, in Chapter 5 we investigate pseudolocality on large hyperbolic balls in dimension two. We obtain results recording how various *almost-hyperbolic* conditions are preserved under Ricci flow, before achieving the main result of this Chapter asserting that if a complete smooth Ricci flow is initially, in a sense made precise in Theorem 5.1.6, *almost-hyperbolic* on a sufficiently large hyperbolic ball, then the Gauss curvature at the central point remains controlled for a time that is exponential in the radius of the ball. Within this chapter we provide some classical PDE regularity theory from [LSU68] that can be applied to the Ricci flow equation in two dimensions. These results are well-known and included for the readers convenience.

Chapter 2

Background Material

In this chapter we fix notation and clarify basic notions that will be used throughout. We also provide the statement of well known classical results that will be used in later chapters.

2.1. Notation

Given a smooth manifold $\mathcal{M}$ (that we shall assume is connected) we denote the space of sections of a vector bundle $E \to \mathcal{M}$ by $\Gamma(E)$. The tangent bundle is denoted by $T\mathcal{M}$ and its dual the cotangent bundle is denoted by $T^*\mathcal{M}$. The space of vector fields on $\mathcal{M}$ is denoted $\Gamma(T\mathcal{M})$ and the space of 1-forms on $\mathcal{M}$ is $\Gamma(T^*\mathcal{M})$. By a (p,q)-tensor we refer to an element of $\Gamma\left(\bigotimes^p T\mathcal{M} \otimes \bigotimes^q T^*\mathcal{M}\right)$, for given $p, q \in \mathbb{N}$.

A Riemannian metric g on $\mathcal{M}$ is an element of $\Gamma\left(\mathrm{Sym}^2_+\left(T^*\mathcal{M}\right)\right)$, i.e. a positive-definite symmetric bilinear form on $\mathcal{M}$. Given any $p \in \mathcal{M}$, we have that g_p is a positive definite symmetric bilinear map $g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$. A metric may be extended to act on elements in $\bigotimes^p T\mathcal{M} \otimes \bigotimes^q T^*\mathcal{M}$ for arbitrary $p, q \in \mathbb{N}$.

A metric g gives an isomorphism from the tangent space $T_p\mathcal{M}$ to its dual space $T_p^*\mathcal{M}$ via the mapping $X \mapsto (Y \mapsto g(Y, X))$. This can be done for any $p \in \mathcal{M}$ and provides a way of transforming vectors to co-vectors and vice versa. It allows a notion of tracing/contracting over any two indices irrespective of their types. This is done by first converting one so they have different types and then contracting by evaluation; i.e. if $X \in T_p\mathcal{M}$ and $\omega \in T_p^*\mathcal{M}$ then we define $\mathrm{tr}(\omega \otimes X) := \omega(X)$.

The Levi-Civita connection of a Riemannian metric g will be denoted by ∇, or sometimes ∇_g to emphasise the metric being referred to, and the same notation will be used to refer to its extension to arbitrary tensor fields. That is it's extension to $\nabla : \Gamma\left(\bigotimes^p T\mathcal{M} \otimes \bigotimes^q T^*\mathcal{M}\right) \to \Gamma\left(\bigotimes^{p+1} T\mathcal{M} \otimes \bigotimes^q T^*\mathcal{M}\right)$. The divergence operator δ_g takes (p,q)-tensors to $(p, q-1)$-

tensors, i.e. gives a mapping $\Gamma\left(\bigotimes^p T\mathcal{M} \otimes \bigotimes^q T^*\mathcal{M}\right) \to \Gamma\left(\bigotimes^p T\mathcal{M} \otimes \bigotimes^{q-1} T^*\mathcal{M}\right)$. It is defined by $\delta_g(T) := -\mathrm{tr}_{12}(\nabla^g T)$, and is the formal adjoint of ∇. This gives rise to the notion of integration by parts. The Laplace-Beltrami operator of g is $\Delta_g := \mathrm{tr}_g[\nabla^g \circ d]$ where d denotes the exterior derivative. For a function $f : \mathcal{M} \to \mathbb{R}$ we have that $\Delta_g f = \mathrm{tr}_g[\mathrm{Hess}(f)]$ where $\mathrm{Hess}(f)$ denotes the symmetric $(0,2)$-tensor $\nabla^g df$.

The Riemann curvature tensor associated to ∇ is the map $\Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \times \Gamma(T\mathcal{M}) \to \Gamma(T\mathcal{M})$ defined by $(X,Y,Z) \mapsto R(X,Y)Z := -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$. This is a $(1,3)$ tensor, i.e. an element of $\Gamma\left(T\mathcal{M} \otimes \bigotimes_{i=1}^3 T^*\mathcal{M}\right)$. The isomorphism between the a tangent space and its dual space (i.e. cotangent space) allows us to view this as a $(0,4)$ tensor field, that is as an element of $\Gamma\left(\bigotimes_{i=1}^4 T^*\mathcal{M}\right)$, denoted by Rm_g and defined by $\mathrm{Rm}_g(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle_g$ where $X,Y,Z,W \in T\mathcal{M}$. Then the Ricci curvature is the tensor defined via $\mathrm{Ric}_g := (\mathrm{tr}_g)_{2,4} \mathrm{Rm}_g = \mathrm{tr}_g[Z \mapsto R(X,Z)Y]$ where the subscript means we contract over the second and fourth indices. It follows that Ric_g is in fact a symmetric $(0,2)$ tensor which means that $\mathrm{Ric}_g \in \Gamma\left(\mathrm{Sym}^2(T^*\mathcal{M})\right)$. Finally the Scalar curvature is the function given by $\mathrm{R}_g := \mathrm{tr}_g \mathrm{Ric}_g$.

For an arbitrary (p,q)-tensor $T \in \Gamma\left(\bigotimes^p T\mathcal{M} \otimes \bigotimes^q T^*\mathcal{M}\right)$ we define its C^k-norm, with respect to a metric g, by $||T||_{C^k(\mathcal{M},g)} := \sum_{i=0}^k \sup_{\mathcal{M}} \left|\nabla^i T\right|_g$. The volume form associated to the metric g will be written as dV_g and is locally defined by $dV_g = \sqrt{\det[g]}dx$. The volume measured with respect to g will be Vol_g and the ball of radius $r > 0$ centred at $p \in \mathcal{M}$ is $\mathbb{B}_g(p,r)$. Throughout we use the abbreviated notations $\mathrm{Vol}\mathbb{B}_g(x,r) := \mathrm{Vol}_g[\mathbb{B}_g(x,r)]$, $\mathrm{Vol}\partial\mathbb{B}_g(x,r) := \mathrm{Vol}_g[\partial\mathbb{B}_g(x,r)]$ and if $K \in \mathbb{R}$ then by $\mathrm{Ric}_g \geq K$ we mean that $\mathrm{Ric}_g \geq Kg$ as bilinear forms.

Suppose $(\mathcal{M}, g)$ is a smooth n-dimensional Riemannian manifold and $p \in \mathcal{M}$. Given any $X \in T_p\mathcal{M}$ we may consider the unique geodesic γ with the initial conditions that $\gamma_X(0) = p$ and $\gamma_X'(0) = X$. Then $\Omega_p := \{X \in T_p\mathcal{M} : \gamma_X(1) \text{ is defined}\}$ is an open subset of $T_p\mathcal{M}$, and the exponential map $\exp_p : \Omega_p \to \mathcal{M}$, defined by $\exp_p(X) := \gamma_X(1)$, is smooth and gives a diffeomorphism onto its image once restricted to a ball $\mathbb{B}(0,\varepsilon) \subset T_p\mathcal{M}$ for some $\varepsilon > 0$. The injectivity radius of $\mathcal{M}$ at a point $p \in \mathcal{M}$, denoted by $\mathrm{inj}_g(p)$, is the largest such $\varepsilon > 0$, i.e. the largest $\varepsilon > 0$ for which $\exp_p$ is a diffeomorphism onto its image once restricted to the ball $\mathbb{B}(0,\varepsilon) \subset T_p\mathcal{M}$. If we consider $\Omega := \cup_{p \in \mathcal{M}} \Omega_p \subset T\mathcal{M}$ we can consider the smooth map $\exp : \Omega \subset T\mathcal{M} \to \mathcal{M}$ given by $(X,p) \mapsto \exp_p(X)$ where $p \in \mathcal{M}$ and $X \in \Omega_p \subset T_p\mathcal{M}$.

An embedded submanifold of $\mathcal{M}$ is a subset $S \subset \mathcal{M}$ that is itself a manifold in the subspace topology, endowed with a smooth structure with respect to which the inclusion map $S \hookrightarrow \mathcal{M}$ is a smooth embedding. Recall that if N is a smooth manifold then $F : N \to \mathcal{M}$ is a smooth embedding if F is a smooth immersion and a homeomorphism onto its image $F(N) \subset \mathcal{M}$ when the image is equipped with the subspace topology inherited from $\mathcal{M}$. Here we say smooth immersion to mean that F is a smooth map and that at every $q \in N$ the differential $dF_q : T_qN \to T_{F(q)}\mathcal{M}$ is injective (equivalently $\mathrm{rank}\, F = \dim(N)$).

Given an embedded submanifold $S \subset \mathcal{M}$ and a point $p \in S$ we can define the normal space to T_pS as $N_pS := (T_pS)^\perp$, i.e. the set of all vectors normal to S at p. The union of all such normal spaces is the normal bundle of S in $\mathcal{M}$ which is denoted by NS. Then we have a normal exponential map $\exp^\perp : \Omega^* \subset NS \to \mathcal{M}$ given by the restriction of the exponential map $\exp$ of $\mathcal{M}$. Therefore, given $V \in N_pS$ the normal exponential map $\exp^\perp$ sends V to $\gamma_V(1)$ provided $\gamma_V(1)$ is defined.

The tubular neighbourhood theorem (Theorem 6.24 in [Lee03] for example) tells us that if S is an embedded submanifold of $\mathcal{M}$ then there exists a tubular neighbourhood U of S. That is, the set U is the diffeomorphic image under $\exp^\perp$ of an open set of the form $\{(V,p) \in NS : |V| < b(p)\}$ for some positive continuous $b : S \to (0,\infty)$, and satisfies $S \subset U \subset \mathcal{M}$. This is not true for immersed submanifolds, as can be seen by considering a *figure-8* submanifold of $\mathbb{R}^2$, for example.

2.2. Volume Comparison

A rough principle in Riemannian geometry is that bounds of the form $\mathrm{Ric}_g \geq (n-1)H$ on a manifold $\mathcal{M}$, for some $H \in \mathbb{R}$, allow us to control the geometry of $\mathcal{M}$ in terms of the geometry of the space of constant sectional curvature H, thus constant Ricci curvature $(n-1)H$, which we denote by $\mathcal{M}_H$. We denote the metric on $\mathcal{M}_H$ by g_H. An example of this phenomenon is the Bishop-Gromov volume comparison theorem which we state below.

Theorem 2.2.1 (Bishop-Gromov volume comparison; Theorem 0.7 in [Che01])**.** *Let $(\mathcal{M}, g)$ denote a complete, smooth n-dimensional Riemannian manifold satisfying $\mathrm{Ric}_g \geq (n-1)H$ throughout $\mathcal{M}$ for some $H \in \mathbb{R}$. Then given any point $p \in \mathcal{M}$, the functions*

$$F(r) := \frac{\mathrm{Vol}\mathbb{B}_g(p,r)}{\mathrm{Vol}\mathbb{B}_{g_H}(p_H,r)} \qquad \textit{and} \qquad f(r) := \frac{\mathrm{Vol}\partial\mathbb{B}_g(p,r)}{\mathrm{Vol}\partial\mathbb{B}_{g_H}(p_H,r)} \tag{2.2.1}$$

are nonincreasing in r for an arbitrary $p_H \in \mathcal{M}_H$.

As $r \downarrow 0$ we have both $F(r) \uparrow 1$ and $f(r) \uparrow 1$. Therefore a particular consequence of (2.2.1) is that

$$\mathrm{Vol}\mathbb{B}_g(p,r) \leq \mathrm{Vol}\mathbb{B}_{g_H}(p_H,r) \qquad \text{and} \qquad \mathrm{Vol}\partial\mathbb{B}_g(p,r) \leq \mathrm{Vol}\partial\mathbb{B}_{g_H}(p_H,r) \tag{2.2.2}$$

for all $r \in (0,\infty)$. Moreover the result of Theorem 2.2.1 is local. That is, if $(\mathcal{M}, g)$ is a smooth n-dimensional Riemannian manifold with $p \in \mathcal{M}$ and $R > 0$ such that $\mathbb{B}_g(p,R) \subset\subset \mathcal{M}$, and we only know that $\mathrm{Ric}_g \geq (n-1)H$ throughout $\mathbb{B}_g(p,R)$ rather than the whole of $\mathcal{M}$, then we may still conclude that the functions defined in (2.2.1) are nonincreasing in r for $r \in (0,R]$.

A useful consequence of Theorem 2.2.1 for our purposes is the following well-known result, which is an explicit statement of ideas already implicit in the early works of Cheeger and Gromov (see [Che01] and [Gro99] for example).

Lemma 2.2.2 (Propagation of lower volume bounds with global Ricci lower bounds)**.** *Let* $(\mathcal{M}, g)$ *be a complete smooth* n*-dimensional Riemannian manifold (without boundary) with a point* $p \in \mathcal{M}$*, and satisfying* $\mathrm{Ric}_g \geq (n-1)H$ *throughout* $\mathcal{M}$ *for some* $H \in \mathbb{R}$*. Assume* $\mathrm{Vol}\mathbb{B}_g(p, 1) \geq v_0 > 0$*. Then there exists a sequence* $v_k > 0$*, defined for* $k \in \mathbb{N}$*, depending only on* n*,* H *and* v_0*, such that for any* $k \in \mathbb{N}$ *and every* $x \in \mathbb{B}_g(p, k)$ *we have* $\mathrm{Vol}\mathbb{B}_g(x, s) \geq v_k s^n$ *for every* $s \in (0, 1]$*.*

Frequently, we will need to pass from a lower bound on the volume of a ball to lower bounds on the volumes of compactly contained sub-balls. That this is possible in the presence of Ricci lower bounds follows from a standard comparison geometry argument, and is the content of the following result. Once again, this makes explicit ideas already implicit in the works of Cheeger and Gromov (see either [Che01] or [Gro99] for example).

Lemma 2.2.3 (Propagation of lower volume bounds with local Ricci lower bounds)**.** *Let* $(\mathcal{M}, g)$ *be a smooth (not necessarily complete)* n*-dimensional Riemannian manifold. Suppose* $p \in \mathcal{M}$ *and* $R > 0$ *such that* $\mathbb{B}_g(p, R) \subset\subset \mathcal{M}$ *and* $\mathrm{Ric}_g \geq (n-1)H$ *throughout* $\mathbb{B}_g(p, R)$ *and* $\mathrm{Vol}\mathbb{B}_g(p, R) \geq v > 0$*. Let* $r \in (0, R)$ *and define* $s := R - r > 0$*. Then there exists a constant* $C > 0$*, depending only on* n*,* H*,* R*,* v *and* s*, such that for any* $x \in \mathbb{B}_g(p, r)$ *we have* $\mathrm{Vol}\mathbb{B}_g(x, s) \geq C$*.*

2.3. Ricci Flow

A Ricci flow solution $g(t)$ on a smooth n-dimensional manifold $\mathcal{M}$, defined for all $t \in [0, T]$, is a one-parameter family of smooth Riemannian metrics $g(t)$, for $t \in [0, T]$, on $\mathcal{M}$ whose evolution is governed by the equation

$$\frac{\partial g}{\partial t}(t) = -2\mathrm{Ric}_{g(t)} \tag{2.3.1}$$

with $g(0) := g_0$ for some given initial metric g_0 on $\mathcal{M}$. The Ricci flow equation in (2.3.1) can be viewed as a non-linear heat equation.

A Ricci flow is complete if for each t the Riemannian manifold $(\mathcal{M}, g(t))$ is complete (i.e. complete as a metric space under the distance function $d_{g(t)}$ induced by $g(t)$). A Ricci flow is of bounded curvature if there is some constant $K > 0$ such that

$$\sup \left\{ |\mathrm{Rm}|_{g(t)}(x) : (x, t) \in \mathcal{M} \times [0, T] \right\} \leq K.$$

Existence and uniqueness of Ricci flow is well-understood for metrics that are complete and of bounded curvature. This may be summarised in the following result.

Theorem 2.3.1 (Ricci flow existence and uniqueness; [Ham82], [Shi89], [DeT03], [Chen06]). *Assume that* $(\mathcal{M}, g_0)$ *is a smooth complete Riemannian manifold of dimension* n *such that for some* $K > 0$ *we have that* $|\mathrm{Rm}|_{g_0} \leq K$ *throughout* $\mathcal{M}$*. Then there exists* $T = T(K, n) > 0$ *and a complete Ricci flow* $(g(t))_{t \in [0,T]}$ *, with* $g(0) = g_0$*, which has bounded curvature. Moreover, any other complete Ricci flow of bounded curvature taking* g_0 *as its initial value must agree with* $g(t)$ *for as long as both flows exist.*

This result is particularly remarkable in dimensions higher than two since the Ricci flow equation is not parabolic in such dimensions. Therefore, the standard theory of quasilinear equations (as found, for example, in [LSU68]) cannot be applied. This difficulty is due to the diffeomorphism invariance of the equation, and is overcome by the so-called DeTurck trick in [DeT03]. A rough overview is that the equation is adjusted in an appropriate way to make it parabolic. The standard parabolic theory is then applied to this altered equation. Finally, the existence for the altered equation is used to deduce existence for the original equation.

In two dimensions existence and uniqueness has been fully settled, irrespective of any completeness or bounded curvature assumptions, by the work of Gregor Giesen and Peter Topping in [GT11] and Peter Topping in [Top15]. Their work yields the following result, which is a combination of Theorem 1.3 in [GT11] (which establishes the existence) and Theorem 1.1 in [Top15] (which establishes the uniqueness).

Theorem 2.3.2 (2D existence and uniqueness; [GT11], [Top15]). *Let* $(\mathcal{M}^2, g_0)$ *be a smooth two-dimensional surface which may be incomplete, and is allowed to have unbounded curvature. Depending on the conformal type of* $(\mathcal{M}, g_0)$*, define* $T \in (0, \infty]$ *by*

$$T := \begin{cases} \frac{1}{8\pi} \operatorname{Vol}_{g_0}(\mathcal{M}) & \textit{if } (\mathcal{M}, g_0) \cong S^2 \\ \frac{1}{4\pi} \operatorname{Vol}_{g_0}(\mathcal{M}) & \textit{if } (\mathcal{M}, g_0) \cong \mathbb{C} \quad \textit{or} \quad (\mathcal{M}, g_0) \cong \mathbb{RP}^2 \\ \quad \infty & \textit{otherwise.} \end{cases} \tag{2.3.2}$$

Then there exists a unique smooth Ricci flow $g(t)$ *on* $\mathcal{M}$*, defined for all* $t \in [0, T)$*, such that*

1. $g(0) = g_0$,
2. $g(t)$ *is* **instantaneously complete***, that is* $(\mathcal{M}, g(t))$ *is complete for every* $t \in (0, T)$*, and*
3. $g(t)$ *is* **maximally stretched***, which is to say that given any other Ricci flow* $\tilde{g}(t)$ *on* $\mathcal{M}$*, defined for all* $t \in [0, \tilde{T}]$ *for some* $\tilde{T} > 0$*, conformally equivalent to* $g(t)$*, with* $\tilde{g}(0) \leq g(0)$*, then* $\tilde{g}(t) \leq g(t)$ *for every* $t \in [0, \min\{T, \tilde{T}\}]$.

In dimensions $n \geq 3$ the problem of well-posedness is less understood. It is not reasonable to expect the generality of Theorem 2.3.2 to carry across to the higher dimensional setting. Indeed, as illustrated in [Top14] for example, we may consider the underlying smooth three-manifold $S^2 \times \mathbb{R}$, endowed with a warped product metric so that metrically it consists of an infinite chain of three-spheres connected by thinner and thinner (and longer and longer) necks. Then given any $\varepsilon > 0$, we may pick a neck that is sufficiently thin and long so that the Ricci flow will pinch it by time ε. A more thorough treatment of this type of neck-pinch singularity may be found in Section 1.3.2 of [Top06], for example. Hence we observe that, in general, we cannot expect there to be any traditional Ricci flow from a general smooth manifold of dimension $n \geq 3$.

However, there are existence results valid in the presence of particular lower curvature bounds or lower volume bounds. For example, Theorem 1.1 in [Hoc16] establishes existence, in dimension three, under global Ricci lower bounds and a global non-collapsed assumption. This global non-collapsed assumption is to require the volume of *every* unit ball is uniformly bounded below. This result has been subsequently improved in [ST17] (see Theorem 1.7) to assert that one may assume a time-independent lower Ricci curvature bound for the flow, along with bi-Hölder estimates on the distance functions at different times.

In the direction of uniqueness, recent work of Brett Kotschwar has obtained uniqueness for complete flows satisfying "quadratic curvature growth" rather than bounded curvature. The rough idea is to examine a certain weighted energy type functional; the functional is essentially the L^2-difference of the curvature tensors with weighted lower order terms included to improve the evolution equation. Full details may be found in [Kot12], with subsequent refinements in [Kot15].

2.4. Simon-Topping Local Ricci Flow

Assume $(\mathcal{M}, g_0, x_0)$ is a complete, smooth, pointed Riemannian three-manifold satisfying, for given $\alpha_0, v_0 > 0$, that $\mathrm{Ric}_{g_0} \geq -\alpha_0$ throughout $\mathcal{M}$ and that $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0$. As we have seen in Section 2.3, it is unreasonable to ask for a traditional smooth Ricci flow solution on $\mathcal{M}$. That is, we cannot expect to find a smooth Ricci flow solution $g(t)$ defined throughout $\mathcal{M}$ for all times $t \in [0, T]$ for some $T > 0$.

To overcome the issues illustrated in Section 2.3, Hochard develops a notion of *Partial Ricci flows* that are only defined on a subset $U \subset\subset \mathcal{M}$, rather than globally throughout the entirety of $\mathcal{M}$, see [Hoc16] for full details.

In their recent works [ST16, ST17], Miles Simon and Peter Topping refine and extend the ideas of Hochard in [Hoc16] to establish that the resulting flow may be assumed to enjoy uniform in time lower Ricci curvature bounds and bi-Hölder estimates between the distance function at different times of the flow. Such flows allow them to locally 'smooth out' the initial data, which is the content of Simon and Topping's *Mollification theorem*, Theorem 1.1 in [ST17] (a variant of which is stated as Theorem 2.4.12 here). Moreover, the improved estimates obtained are essential to their results in [ST17]. Most notably, their resolution of the three-dimensional conjecture of Anderson-Cheeger-Colding-Tian regarding the regularity of noncollapsed Ricci limit spaces.

We will provide a brief overview of their techniques within this section. Our starting point is the *local existence theorem* from [ST17], establishing that it is possible to locally run the Ricci flow from a complete, smooth, pointed Riemannian three-manifold satisfying both the Ricci curvature bound and noncollapsed condition stated above.

Theorem 2.4.1 (Local existence theorem; Theorem 1.6 in [ST17])**.** *Suppose $s_0 \geq 4$. Suppose $\left(\mathcal{M}^3, g_0\right)$ is a Riemannian manifold, $x_0 \in \mathcal{M}$ for which $\mathbb{B}_{g_0}(x_0, s_0) \subset\subset \mathcal{M}$ and $\mathrm{Ric}_{g_0} \geq -\alpha_0$ on $\mathbb{B}_{g_0}(x_0, s_0)$ and $\mathrm{Vol}\mathbb{B}_{g_0}(x, 1) \geq v_0 > 0$ for all $x \in \mathbb{B}_{g_0}(x_0, s_0 - 1)$. Then there exist constants $T = T(\alpha_0, v_0) > 0, \alpha = \alpha(\alpha_0, v_0) > 0, c_0 = c_0(\alpha_0, v_0) > 0$ and a Ricci flow $g(t)$ defined for $0 \leq t \leq T$ on $\mathbb{B}_{g_0}(x_0, s_0 - 2)$, with $g(0) = g_0$ where defined, such that for all $0 < t \leq T$ we have*

$$\mathrm{Ric}_{g(t)} \geq -\alpha \qquad \textit{and} \qquad |\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t} \qquad \textit{throughout} \qquad \mathbb{B}_{g_0}(x_0, s_0 - 2).$$

In the first part of this section we outline how Theorem 2.4.1 is obtained in [ST17], whilst simultaneously recording several further results from [ST16, ST17] that we will later require.

The first major step of the proof is to conformally modify the metric g_0 to make it complete on the ball $\mathbb{B}_{g_0}(x_0, s_0)$, whilst remaining unchanged throughout $\mathbb{B}_{g_0}(x_0, s_0 - 1)$. That this may be done is a result of Hochard's Lemma 6.2 in [Hoc16], though we state the scaled form appearing as Lemma 4.3 in [ST17] below. Moreover, whilst being undertaken by Hochard within [Hoc16], the idea of cutting off a metric locally and replacing it with a complete hyperbolic metric in order to start the flow originates in earlier work of Peter Topping, see [Top12].

Lemma 2.4.2 (Conformal alteration; Lemma 6.2 in [Hoc16] and Lemma 4.3 in [ST17])**.** *Let (N^n, g) be a smooth, possibly incomplete, Riemannian manifold with $U \subset N$ open. Assume that for some $\rho \in (0, 1]$ we have that $|\mathrm{Rm}|_g \leq \rho^{-2}$ throughout U and that for every $x \in U$ we have both that $\mathbb{B}_g(x, \rho) \subset\subset N$ and $\mathrm{inj}_g(x) \geq \rho$. Then there exists a constant $\gamma = \gamma(n) \geq 1$, an open set $\tilde{U} \subset U$ and a smooth Riemannian metric $\tilde{g}$, defined throughout $\tilde{U}$, such that every connected component of $(\tilde{U}, \tilde{g})$ is a complete Riemannian manifold satisfying*

1. *$|\mathrm{Rm}|_{\tilde{g}} \leq \gamma\rho^{-2}$ throughout $\tilde{U}$,*
2. *$U_\rho \subset \tilde{U} \subset U$, and*
3. *$\tilde{g} = g$ throughout $\tilde{U}_\rho \supset U_{2\rho}$*

where we use the notation that for $s > 0$ we define $U_s := \{x \in U : \mathbb{B}_g(x, s) \subset\subset U\}$.

Smoothness of g_0, and that $\mathbb{B}_{g_0}(x_0, s_0) \subset\subset \mathcal{M}$, ensure that for suitably small $\rho \in (0, 1]$ we have $|\mathrm{Rm}|_{g_0} \leq \rho^{-2}$ throughout $\mathbb{B}_{g_0}(x_0, s_0)$. Therefore we may apply Lemma 2.4.2 to conformally alter g_0 to make it complete on $\mathbb{B}_{g_0}(x_0, s_0)$, whilst remaining unchanged on $\mathbb{B}_{g_0}(x_0, s_0 - 1)$. We may now appeal to Shi's existence theorem, i.e. Theorem 2.3.1, to obtain a smooth Ricci flow solution $g(t)$, which is, in particular, defined throughout $\mathbb{B}_{g_0}(x_0, s_0 - 1)$ for all times $t \in [0, T]$ for some $T > 0$, satisfying that $g(0) = g_0$ throughout $\mathbb{B}_{g_0}(x_0, s_0 - 1)$, and moreover with $|\mathrm{Rm}|_{g(t)} \leq \frac{C}{t}$ throughout $\mathbb{B}_{g_0}(x_0, s_0 - 1) \times (0, T]$. The problem that remains to be overcome is the bad dependencies, namely that both T and C above depend on ρ, which means they depend on the particular manifold rather than only depending on the initial Ricci lower bound and degree of noncollapsedness.

Improving these dependencies requires machinery developed in both [ST16, ST17]. We begin by stating the so-called *Double Bootstrap* result from [ST16] which establishes that, in the presence of local C/t curvature decay, local pointwise lower Ricci bounds propagate forwards in time under Ricci flow for a definite amount of time. The precise result is the following.

Lemma 2.4.3 (Double bootstrap; Lemma 4.2 in [ST17] or Lemma 9.1 in [ST16])**.** *Let $\left(\mathcal{M}^3, g(t)\right)$ be a smooth Ricci flow, for $0 \leq t \leq T$, such that for some $x \in \mathcal{M}$ we have $\mathbb{B}_{g(0)}(x, 2) \subset\subset \mathcal{M}$, and so that $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ on $\mathbb{B}_{g(0)}(x, 2) \times (0, T]$ for some $c_0 \geq 1$ and $\mathrm{Ric}_{g(0)} \geq -\delta_0$ on $\mathbb{B}_{g(0)}(x, 2)$ for some $\delta_0 > 0$. Then there exists $S = S(c_0, \delta_0) > 0$ such that for all $0 < t \leq \min\{T, S\}$ we have*

$$\mathrm{Ric}_{g(t)}(x) \geq -100\delta_0 c_0. \tag{2.4.1}$$

Further, we will frequently require the *Local lemma*, Lemma 4.1 in [ST17], which roughly establishes that if a three-dimensional Ricci flow enjoys local time-independent pointwise Ricci lower bounds, and an initial noncollapsed condition, then it enjoys C/t curvature decay for some definite amount of time. The precise result is the following.

Lemma 2.4.4 (The local lemma 4.1 in [ST17])**.** *Given any $v_0 > 0$ there exists $C_0 = C_0(v_0) \geq 1$ such that the following is true. Let $(N^3, g(t))$ be a smooth three dimensional Ricci flow, for $0 \leq t \leq T$, such that for a fixed $x \in N$ we have that $\mathbb{B}_{g(t)}(x, 1) \subset\subset N$ for each $t \in [0, T]$. Further assume that $\mathrm{Vol}\mathbb{B}_{g(0)}(x, 1) \geq v_0 > 0$ and that $\mathrm{Ric}_{g(t)} \geq -1$ throughout $\mathbb{B}_{g(t)}(x, 1)$ for each $t \in [0, T]$. Then there is a constant $\hat{T} = \hat{T}(v_0) > 0$ such that for all $0 < t \leq \min\left\{T, \hat{T}\right\}$ we have both*

$$|\mathrm{Rm}|_{g(t)}(x) \leq \frac{C_0}{t} \quad \text{and} \quad \mathrm{inj}_{g(t)}(x) \geq \sqrt{\frac{t}{C_0}}. \tag{2.4.2}$$

Frequently, it will be more convenient to appeal to a scaled variant of this result. In particular, we record the following scaled variant, where we have weakened the required Ricci lower bound to $-\gamma$ rather than -1. Lemma 4.1 in [ST17] corresponds to the $\gamma = 1$ case. The same statement is actually given as Lemma 2.1 in [ST16], but with less good dependencies given for the curvature estimates achieved. The following result makes explicit ideas that are implicit in [ST16] and

[ST17], and appears as Lemma A.1 in [MT18].

Lemma 2.4.5 (Variant of the local lemma 4.1 in [ST17]; Lemma A.1 in [MT18])**.** *Given any $v_0 > 0$, there exists $C_0 = C_0(v_0) \geq 1$ such that the following is true. Let $\left(\mathcal{M}^3, g(t)\right)$, for $0 \leq t \leq T$, be a smooth Ricci flow such that for some fixed $x \in \mathcal{M}$ we have $\mathbb{B}_{g(t)}(x,1) \subset\subset \mathcal{M}$ for all $0 \leq t \leq T$, and so that for any $0 < r \leq 1$, $\mathrm{Vol}\mathbb{B}_{g(0)}(x,r) \geq v_0 r^3 > 0$ and $\mathrm{Ric}_{g(t)} \geq -\gamma$ on $\mathbb{B}_{g(t)}(x,1)$ for some $\gamma > 0$ and all $0 \leq t \leq T$. Then there exists $S = S(v_0,\gamma) > 0$ such that for all $0 < t \leq \min\{T,S\}$ we have both*

$$|\mathrm{Rm}|_{g(t)}(x) \leq \frac{C_0}{t} \quad \textit{and} \quad \mathrm{inj}_{g(t)}(x) \geq \sqrt{\frac{t}{C_0}}. \tag{2.4.3}$$

Proof. Without loss of generality we assume that $\gamma \geq 1$; if $0 < \gamma < 1$ then we could replace γ by 1 since $\mathrm{Ric}_{g(t)} \geq -\gamma$ would give that $\mathrm{Ric}_{g(t)} \geq -1$. Then consider the rescaled flow $g_p(t) := \gamma g\left(\frac{t}{\gamma}\right)$ for times $0 \leq t \leq \gamma T$. We first observe that

$$\mathrm{Vol}\mathbb{B}_{g_p(0)}(x,1) = \gamma^{\frac{3}{2}}\,\mathrm{Vol}\mathbb{B}_{g(0)}\left(x, \frac{1}{\sqrt{\gamma}}\right) \geq \gamma^{\frac{3}{2}}\gamma^{-\frac{3}{2}} v_0 = v_0. \tag{2.4.4}$$

Moreover, for any $0 \leq t \leq \gamma T$ we have both

$$\mathbb{B}_{g_p(t)}(x,1) = \mathbb{B}_{g\left(\frac{t}{\gamma}\right)}\left(x, \frac{1}{\sqrt{\gamma}}\right) \subset\subset \mathcal{M} \tag{2.4.5}$$

and for any $z \in \mathbb{B}_{g_p(t)}(x,1)$

$$\mathrm{Ric}_{g_p(t)}(z) = \mathrm{Ric}_{\gamma g\left(\frac{t}{\gamma}\right)}(z) \geq -1 \tag{2.4.6}$$

since $z \in \mathbb{B}_{g\left(\frac{t}{\gamma}\right)}(x,1)$. Therefore, by combining (2.4.4), (2.4.5) and (2.4.6) we have the hypotheses to be able to apply Lemma 4.1 from [ST17], i.e. Lemma 2.4.4 above. Doing so gives us constants $C_0 = C_0(v_0) \geq 1$ and $S_0 = S_0(v_0) > 0$ such that for all $0 < t \leq \min\{\gamma T, S_0\}$ we have both

$$|\mathrm{Rm}|_{g_p(t)}(x) \leq \frac{C_0}{t} \quad \text{and} \quad \mathrm{inj}_{g_p(t)}(x) \geq \sqrt{\frac{t}{C_0}}. \tag{2.4.7}$$

Both the estimates in (2.4.7) are preserved under rescaling back to the original flow $g(t)$. Then, by taking $S := \frac{S_0}{\gamma} > 0$, which does indeed depend only on v_0 and γ, we deduce (2.4.7) for the flow $g(t)$ itself and for all times $0 < t \leq \min\{T,S\}$. ∎

In order to appeal to these results successively, we require being able to control how distances are changing over time under the flow. That is, Lemma 2.4.3 requires compactness of a time 0 ball whilst Lemma 2.4.5 requires compactness of balls at times $t > 0$. Therefore we need to be able to control balls at a given time of the flow with respect to balls at a different time of the flow. Such control is provided by the following two results, the first is the *shrinking balls lemma* (see Corollary 3.3 in [ST16]) whilst the second is the *expanding balls lemma* (see Lemma 3.1 in [ST16] and Lemma 2.1 [ST17]).

Lemma 2.4.6 (The shrinking balls lemma; Corollary 3.3 in [ST16])**.** *Suppose $(\mathcal{M}^n, g(t))$ is a Ricci flow for $0 \leq t \leq T$ on a smooth n-manifold $\mathcal{M}$. Then there exists a $\beta = \beta(n) \geq 1$ such that the following is true. Suppose $x_0 \in \mathcal{M}$ and that $\mathbb{B}_{g(0)}(x_0, r) \subset\subset \mathcal{M}$ for some $r > 0$, and $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$, or more generally $\mathrm{Ric}_{g(t)} \leq (n-1)\frac{c_0}{t}$, on $\mathbb{B}_{g(0)}(x_0, r) \cap \mathbb{B}_{g(t)}(x_0, r - \beta\sqrt{c_0 t})$ for each $t \in (0, T]$ and some $c_0 > 0$. Then for all $0 \leq t \leq T$*

$$\mathbb{B}_{g(t)}\left(x_0, r - \beta\sqrt{c_0 t}\right) \subset \mathbb{B}_{g(0)}(x_0, r). \tag{2.4.8}$$

More generally, for $0 \leq s \leq t \leq T$, we have

$$\mathbb{B}_{g(t)}\left(x_0, r - \beta\sqrt{c_0 t}\right) \subset \mathbb{B}_{g(s)}\left(x_0, r - \beta\sqrt{c_0 s}\right). \tag{2.4.9}$$

Lemma 2.4.7 (The expanding balls lemma; Lemma 3.1 in [ST16] and Lemma 2.1 in [ST17])**.** *Suppose $K > 0$ and $(\mathcal{M}^n, g(t))$ is a Ricci flow for $t \in [-T, 0]$, $T > 0$, on a smooth n-manifold $\mathcal{M}$. Suppose $x_0 \in \mathcal{M}$ and that $\mathbb{B}_{g(0)}(x_0, R) \subset\subset \mathcal{M}$ and $\mathrm{Ric}_{g(t)} \geq -K$ on $\mathbb{B}_{g(0)}(x_0, R) \cap \mathbb{B}_{g(t)}\left(x_0, Re^{Kt}\right) \subset \mathbb{B}_{g(t)}(x_0, R)$ for each $t \in [-T, 0]$. Then for all $t \in [-T, 0]$*

$$\mathbb{B}_{g(t)}\left(x_0, Re^{Kt}\right) \subset \mathbb{B}_{g(0)}\left(x_0, R\right). \tag{2.4.10}$$

Returning to our outline of how to prove Theorem 2.4.1, recall that we have thus far obtained a Ricci flow $g(t)$ throughout $\mathbb{B}_{g_0}(x_0, s_0 - 1) \times [0, T]$, satisfying C/t curvature decay for some $C > 0$ with poor dependencies (i.e. depends on the particular initial manifold). We now improve the curvature estimates for $g(t)$ to have the dependencies required in Theorem 2.4.1. Roughly, we first appeal to the *Double Bootstrap* Lemma 2.4.3 to obtain uniform in time Ricci lower bounds for the flow $g(t)$. Assuming $T > 0$ is suitably reduced, the *expanding balls* lemma 2.4.7 allows us to conclude the required compact inclusions at times $t > 0$ in order to subsequently apply Lemma 2.4.5, using the uniform in time lower Ricci bounds the *double bootstrap* provided, to obtain the required C_0/t curvature decay for the flow $g(t)$. In turn, this improved C_0/t curvature decay now allows us to appeal to Lemma 2.4.3 once again, but this time conclude uniform in time Ricci lower bounds for the flow $g(t)$ with the dependencies required in Theorem 2.4.1.

It only remains to establish that the flow's existence time T can be controlled from below in terms of the dependencies required in Theorem 2.4.1. This is achieved by the *extension lemma*,

Lemma 4.4, in [ST17], which asserts that, after restricting to a strictly smaller spatial ball, we may conclude that the flow $g(t)$ enjoys the same curvature estimates over a controllably longer time interval. The precise result is the following.

Lemma 2.4.8 (Extension Lemma; Lemma 4.4 in [ST17])**.** *Let $v_0 > 0$. Then there exists $c_0 \geq 1$ and $\tau > 0$ for which the following is true. Let $r_1 \geq 2$ and $(\mathcal{M}^3, g_0)$ be a smooth three-dimensional Riemannian manifold such that $\mathbb{B}_{g_0}(x_0, r_1) \subset\subset \mathcal{M}$, $\mathrm{Ric}_{g_0} \geq -\alpha_0$ throughout $\mathbb{B}_{g_0}(x_0, r_1)$ for some $\alpha_0 \geq 1$, and that for any $r \in [0,1]$ and any $x \in \mathbb{B}_{g_0}(x_0, r_1 - r)$ we have $\mathrm{Vol}\mathbb{B}_{g_0}(x, r) \geq v_0 r^3$.*

Further suppose that $g(t)$ is a smooth Ricci flow defined throughout $\mathbb{B}_{g_0}(x_0, r_1)$, for all $t \in [0, l_1]$ with $l_1 \leq \frac{\tau}{200\alpha_0 c_0}$, with $g(0) = g_0$ throughout $\mathbb{B}_{g_0}(x_0, r_1)$, and satisfying that

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\frac{\tau}{l_1} & \text{on } \mathbb{B}_{g_0}(x_0, r_1) \times (0, l_1], \\ |\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t} & \text{on } \mathbb{B}_{g_0}(x_0, r_1) \times (0, l_1]. \end{cases} \tag{2.4.11}$$

Then, setting $l_2 := l_1\left(1 + \frac{1}{4c_0}\right)$ and $r_2 := r_1 - 6\sqrt{\frac{l_2}{\tau}} \geq 1$, the Ricci flow $g(t)$ can be extended smoothly to be defined throughout $\mathbb{B}_{g_0}(x_0, r_2)$, for all times $t \in [0, l_2]$, with

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\frac{\tau}{l_2} & \text{on } \mathbb{B}_{g_0}(x_0, r_2) \times (0, l_2], \\ |\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t} & \text{on } \mathbb{B}_{g_0}(x_0, r_2) \times (0, l_2]. \end{cases} \tag{2.4.12}$$

The idea behind Lemma 2.4.8 is to use Lemma 2.4.2 at time $t = l_1$ to make $g(l_1)$ complete on a superset of $\mathbb{B}_{g_0}\left(x_0, r_1 - 4\sqrt{\frac{l_1}{\tau}}\right)$, whilst remaining unchanged throughout $\mathbb{B}_{g_0}\left(x_0, r_1 - 4\sqrt{\frac{l_1}{\tau}}\right)$. Then Shi's existence theorem (Theorem 2.3.1) may be used to provide the desired extension, with the *doubling-time* estimates of Lemma 2.5.1 and the *double bootstrap* (Lemma 2.4.3) providing the desired curvature control.

Returning to the flow $g(t)$ we have constructed in our outline of the strategy behind Theorem 2.4.1, we can iteratively apply Lemma 2.4.8 to deduce that, after restricting to $\mathbb{B}_{g_0}(x_0, s_0 - 2)$, the flow's existence time T can be taken to have the dependencies required in Theorem 2.4.1. To do this rigorously requires a careful choice of constants to ensure that we cannot lose too much spatial radius before being able to conclude that the time up to which Lemma 2.4.8 provides control has become sufficiently large. The precise details achieving this can be found in Section 4 of [ST17].

For our purposes, we record further results of Simon and Topping that will be useful later. The first records how initial local lower volume bounds propagate forwards in time under local Ricci flow for a definite amount of time. The following is Lemma 2.3 in [ST16].

Lemma 2.4.9 (Lower volume control Lemma 2.3 in [ST16])**.** *Let $(N^n, g(t))$ be a smooth n-dimensional Ricci flow for $t \in [0, T)$, such that $\mathbb{B}_{g(t)}(x_0, \gamma) \subset\subset N$ for some $x_0 \in N$ and $\gamma > 0$, and all $t \in [0, T)$. Assume further that*

- $\mathrm{Ric}_{g(t)} \geq -K$, *for some* $K > 0$, *throughout* $\mathbb{B}_{g(t)}(x_0, \gamma)$ *for each* $t \in [0, T)$,

- $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *throughout* $\mathbb{B}_{g(t)}(x_0, \gamma)$ *for each* $t \in (0, T)$, *with* $c_0 \in (0, \infty)$,

- $\mathrm{Vol}\mathbb{B}_{g(0)}(x_0, \gamma) \geq v_0 > 0$.

Then there exist constants $\varepsilon_0 = \varepsilon_0(v_0, K, \gamma, n) > 0$ *and* $\hat{T} = \hat{T}(v_0, c_0, K, \gamma, n) > 0$ *such that* $\mathrm{Vol}\mathbb{B}_{g(t)}(x_0, \gamma) \geq \varepsilon_0$ *for all* $t \in \left[0, \hat{T}\right] \cap [0, T)$.

The following minor variant of Lemma 2.3 in [ST16], which may also be found as Lemma A.4 in [MT18], will sometimes be a more convenient form for our purposes. We replace the required compactness of a time t ball by compactness of a time 0 ball. Moreover, we now obtain volume estimates for unit balls within a later time t ball, rather than just for a single fixed unit ball at later times t. Again this makes explicit ideas implicitly used in both [ST16] and [ST17].

Lemma 2.4.10 (Variant of lower volume control lemma 2.3 in [ST16]; Lemma A.4 in [MT18])**.** *Suppose that* $(\mathcal{M}^n, g(t))$ *is a smooth Ricci flow over the time interval* $t \in [0, T)$ *and that for some* $R \geq 2$ *we have that* $\mathbb{B}_{g(0)}(x_0, R) \subset\subset \mathcal{M}$ *for some* $x_0 \in \mathcal{M}$*. Moreover assume that*

- $\mathrm{Ric}_{g(t)} \geq -K$ *on* $\mathbb{B}_{g(0)}(x_0, R)$, *for some* $K > 0$ *and all* $t \in [0, T)$,

- $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ *on* $\mathbb{B}_{g(0)}(x_0, R)$, *for some* $c_0 > 0$ *and all* $t \in (0, T)$,

- $\mathrm{Vol}\mathbb{B}_{g(0)}(x_0, 1) \geq v_0 > 0$.

Then there exists $\varepsilon_R = \varepsilon_R\,(v_0, K, R, n) > 0$ *and* $\hat{T} = \hat{T}\,(v_0, c_0, K, n, R) > 0$ *such that for all* $t \in [0, T) \cap \left[0, \hat{T}\right)$ *we have* $\mathbb{B}_{g(t)}(x_0, R-1) \subset \mathbb{B}_{g(0)}(x_0, R)$, *and that for all* $x \in \mathbb{B}_{g(t)}(x_0, R-2)$, *we have* $\mathrm{Vol}\mathbb{B}_{g(t)}(x, 1) \geq \varepsilon_R$.

Proof. Lemma 2.4.6 yields a $\beta = \beta(n) \geq 1$ for which $\mathbb{B}_{g(0)}(x_0, R) \supset \mathbb{B}_{g(t)}(x_0, R - \beta\sqrt{c_0 t})$ for all $t \in [0, T)$. Therefore, for $0 \leq t \leq \min\left\{T, \frac{1}{\beta^2 c_0}\right\}$ we have $\mathbb{B}_{g(t)}(x_0, R-1) \subset \mathbb{B}_{g(0)}(x_0, R)$, so $\mathbb{B}_{g(t)}(x_0, R-1) \subset\subset M$ and the assumed curvature estimates hold on $\mathbb{B}_{g(t)}(x_0, R-1)$ for all such times t. Lemma 2.4.9 above, with $\gamma = 1$ yields $\varepsilon_0 = \varepsilon_0\,(v_0, K, n) > 0$ and $\tilde{T} = \tilde{T}\,(v_0, c_0, K, n) > 0$ such that $\mathrm{Vol}\mathbb{B}_{g(t)}(x_0, 1) \geq \varepsilon_0 > 0$ for all times $0 \leq t \leq \min\left\{T, \frac{1}{\beta^2 c_0}, \tilde{T}\right\}$. Set $\hat{T} := \min\left\{\tilde{T}, \frac{1}{\beta^2 c_0}\right\} > 0$, which depends only on v_0, K, c_0, n and R. Given any $t \in [0, \min\{T, \hat{T}\}]$, the Ricci lower bound $\mathrm{Ric}_{g(t)} \geq -K$ throughout $\mathbb{B}_{g(t)}\,(x_0, R-1)$ allows us, via Bishop-Gromov, to reduce ε_0 to a constant $\varepsilon_R = \varepsilon_R\,(v_0, K, n, R) > 0$ such that for all $x \in \mathbb{B}_{g(t)}\,(x_0, R-2)$, we have $\mathrm{Vol}\mathbb{B}_{g(t)}(x, 1) \geq \varepsilon_R > 0$. ∎

Next, we record the local bi-Hölder estimates obtained for the distance function under local Ricci flow by Simon and Topping. The estimates in (2.4.14) below localise the corresponding global estimates achieved by Simon in [Sim12], whilst the estimates in (2.4.15) are an improvement obtained in [ST17]; in particular, these estimates are reliant upon the time-independent Ricci lower bound enjoyed by the flow.

Lemma 2.4.11 (Bi-Hölder Distance Estimates; Lemma 3.1 in [ST17])**.** *Suppose $(\mathcal{M}^n, g(t))$ is a Ricci flow for $t \in (0,T]$, not necessarily complete, and $r > 0$ is such that for some $x_0 \in \mathcal{M}$, and all $t \in (0,T]$, we have $\mathbb{B}_{g(t)}(x_0, 2r) \subset\subset \mathcal{M}$. Suppose further that for some $c_0, \alpha > 0$, and for each $t \in (0,T]$, we have*

$$-\alpha \leq \mathrm{Ric}_{g(t)} \leq (n-1)\frac{c_0}{t} \tag{2.4.13}$$

throughout $\mathbb{B}_{g(t)}(x_0, 2r)$. Define $\Omega_T := \bigcap_{0<t\leq T} \mathbb{B}_{g(t)}(x_0, r)$. Then for any $x, y \in \Omega_T$ the distance $d_{g(t)}(x,y)$ is unambiguous for all $t \in (0,T]$ and must be realised by a minimising geodesic lying within $\mathbb{B}_{g(t)}(x_0, 2r)$. Then, for any $0 < t_1 \leq t_2 \leq T$, we have

$$d_{g(t_1)}(x,y) - \beta\sqrt{c_0}\left(\sqrt{t_2} - \sqrt{t_1}\right) \leq d_{g(t_2)}(x,y) \leq e^{\alpha(t_2 - t_1)} d_{g(t_1)}(x,y), \tag{2.4.14}$$

where $\beta = \beta(n) > 0$. In particular, $d_{g(t)}$ converges uniformly to a distance metric d_0 on Ω_T as $t \downarrow 0$, and

$$d_0(x,y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x,y) \leq e^{\alpha t} d_0(x,y), \tag{2.4.15}$$

for all $t \in (0,T]$. Moreover, there exists $\gamma > 0$, depending only on n, c_0 and upper bounds for T and r, such that

$$\gamma \left[d_0(x,y)\right]^{1+2(n-1)c_0} \leq d_{g(t)}(x,y) \tag{2.4.16}$$

for all $t \in (0,T]$. Finally, for all $t \in (0,T]$ and $R < R_0 := re^{-\alpha T} - \beta\sqrt{c_0 T} < r$, we have

$$\mathbb{B}_{g(t)}(x_0, R_0) \subset \Omega_T \quad \text{and} \quad \mathbb{B}_{d_0}(x_0, R) \subset\subset \mathcal{O} \tag{2.4.17}$$

where $\mathcal{O}$ is the component of $\mathrm{Interior}\,(\Omega_T)$ *containing x_0.*

Combining all the results from [ST16, ST17] that we have presented in this section allows one to obtain the *Mollification theorem* of Miles Simon and Peter Topping, Theorem 1.1 in [ST17], which is central to obtaining the bi-Hölder correspondence between three-dimensional Ricci limit spaces and topological manifolds in Corollary 1.5 of [ST17], and which appears as Theorem 2.9.2 later. This result rigorously localises the global mollification results achieved by Simon in [Sim12] under the globally noncollapsed regime that the volume of *every* unit ball is uniformly controlled from below.

Theorem 2.4.12 (Mollification Theorem; Variant of Theorem 1.1 in [ST17])**.** *Assume $\alpha_0, v_0 > 0$. Let $(\mathcal{M}^3, g_0)$ be a smooth Riemannian three-manifold with $x_0 \in \mathcal{M}$ such that $\mathbb{B}_{g_0}(x_0, 1) \subset\subset \mathcal{M}$ and with $\mathrm{Ric}_{g_0} \geq -\alpha_0$ throughout $\mathbb{B}_{g_0}(x_0,1)$ and $\mathrm{Vol}\mathbb{B}_{g_0}(x_0,1) \geq v_0$. Then for any $\varepsilon \in$*

$(0, 1/10)$ *there exist positive constants* T, v, α, *and* c_0, *all depending only on* α_0, v_0 *and* ε, *and there is a smooth Ricci flow* $g(t)$ *defined throughout* $\mathcal{B} := \mathbb{B}_{g_0}(x_0, 1-\varepsilon)$ *for all* $t \in [0,T]$, *with* $g(0) = g_0$ *on* $\mathcal{B}$, *such that for each* $t \in [0,T]$ *we have* $\mathbb{B}_{g(t)}(x_0, 1-2\varepsilon) \subset\subset \mathcal{B}$, *and satisfying*

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha & \text{on } \mathcal{B} \times [0,T], \\ |\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t} & \text{on } \mathcal{B} \times (0,T]. \end{cases}$$

Moreover, for all $t \in [0,T]$ *we have that* $\mathrm{Vol}\mathbb{B}_{g(t)}(x_0, 1-2\varepsilon) \geq v$.

Further, if we fix a time $s \in [0,T]$ *and consider* $x, y \in \mathbb{B}_{g(s)}\left(x_0, \frac{1}{2} - 2\varepsilon\right)$, *then* $x, y \in \mathbb{B}_{g(t)}\left(x_0, \frac{1}{2} - \varepsilon\right)$ *for all* $t \in [0,T]$ *so the distance* $d_{g(t)}(x,y)$ *within* $\mathcal{B}$ *is realised by a geodesic within* $\mathbb{B}_{g(t)}(x_0, 1-2\varepsilon) \subset \mathcal{B}$ *where the Ricci flow is defined. Finally, for any* $t \in [0,T]$ *and* x, y *as above, we have*

$$d_{g_0}(x,y) - \beta\sqrt{c_0 t} \leq d_{g(t)}(x,y) \leq e^{\alpha t} d_{g_0}(x,y) \qquad \text{and} \qquad d_{g_0}(x,y) \leq \gamma \left[d_{g(t)}(x,y)\right]^{\frac{1}{1+4c_0}}$$

where $\beta \geq 1$ *is universal and* $\gamma = \gamma(c_0) \in (0,\infty)$, *i.e.* γ *depends only on* α_0, v_0 *and* ε.

2.5. Shi's Derivative Estimates

Under complete Ricci flows of bounded curvature, the global maximum of the curvature cannot instantly rapidly increase. This is a consequence of the *Doubling time estimate* arising in the following result.

Lemma 2.5.1 (Doubling time estimate; Lemma 6.1 in [Cho06]). *Suppose that* $(\mathcal{M}^n, g(t))$ *is a complete Ricci flow solution of bounded curvature defined for all* $t \in [0,T]$, *for some* $T > 0$, *on a smooth* n*-manifold* $\mathcal{M}$. *Assume that* $|\mathrm{Rm}|_{g(0)} \leq K$ *throughout* $\mathcal{M}$. *Then* $|\mathrm{Rm}|_{g(t)} \leq 2K$ *throughout* $\mathcal{M}$ *for all times* $t \in [0,\tau]$, *where* $\tau := \min\left\{T, \frac{1}{16K}\right\} > 0$.

The global Bernstein-Bando-Shi estimates establish that if the curvature of a complete Ricci flow is globally bounded over some definite time interval, then the derivatives of the curvature become bounded for positive times. This is the content of the following result.

Theorem 2.5.2 (Bernstein-Bando-Shi global derivative estimates; Theorem 6.6 in [Cho06]). *Suppose that* $(\mathcal{M}^n, g(t))$ *is a complete Ricci flow solution of bounded curvature on a smooth* n*-manifold* $\mathcal{M}$, *defined for all* $t \in [0,T]$ *for some* $T > 0$. *Assume* $K, \alpha > 0$ *are such that* $|\mathrm{Rm}|_{g(t)} \leq K$ *throughout* $\mathcal{M}$ *for all* $t \in [0,\tau]$, *where* $\tau := \min\left\{T, \frac{\alpha}{K}\right\} > 0$. *Then for every* $l \in \mathbb{N}$ *there exists a constant* $C = C(l,n,\alpha) > 0$ *such that throughout* $\mathcal{M} \times (0,\tau]$ *we have*

$$|\nabla^l \mathrm{Rm}|_{g(t)} \leq \frac{CK}{t^{\frac{l}{2}}}. \tag{2.5.1}$$

These estimates have been localised by W. X. Shi. A useful variant of Shi's derivative estimates is Theorem 14.14 in [Cho08]; the statement is the following

Theorem 2.5.3 (Theorem 14.14 in [Cho08]). *Let $\alpha, K, r > 0$ and $l, n \in \mathbb{N}$. Suppose $\mathcal{M}$ is a smooth n-dimensional manifold with $p \in \mathcal{M}$ and $\mathcal{U}$ an open neighbourhood of p. Assume that $g(t)$ is a smooth Ricci flow solution, defined throughout $\mathcal{U} \times \left[0, \frac{\alpha}{K}\right]$, for which $\overline{\mathbb{B}_{g(0)}(p,r)} \subset\subset \mathcal{U}$, and satisfying that $|\mathrm{Rm}|_{g(t)} \leq K$ throughout $\mathcal{U} \times \left[0, \frac{\alpha}{K}\right]$. Then there exists a constant $C = C(\alpha, K, r, l, n) > 0$ such that*

$$|\nabla^l \mathrm{Rm}|_{g(t)}(z) \leq \frac{C}{t^{\frac{l}{2}}} \tag{2.5.2}$$

throughout $\mathbb{B}_{g(0)}\left(p, \frac{r}{2}\right) \times \left(0, \frac{\alpha}{K}\right]$.

Suppose that a smooth Ricci flow $g(t)$ satisfies the hypotheses of Theorem 2.5.3 with $K \geq 1$. Consider the rescaled flow $\tilde{g}(t) := Kg\left(\frac{t}{K}\right)$ so that on $\mathcal{U} \times [0, \alpha]$ we have $|\mathrm{Rm}|_{\tilde{g}(t)} \leq 1$. Then, since $K \geq 1$, we have that $\overline{\mathbb{B}_{\tilde{g}(0)}(p,r)} \subset \overline{\mathbb{B}_{\tilde{g}(0)}(p, r\sqrt{K})} = \overline{\mathbb{B}_{g(0)}(p,r)} \subset\subset U$. A consequence of Theorem 2.5.3 is that we may conclude that there is a constant $C = C(\alpha, r, l, n) > 0$ such that, at the point p, we have $\left|\nabla^l \mathrm{Rm}\right|_{\tilde{g}(t)}(p) \leq Ct^{-\frac{l}{2}}$ for all $t \in (0, \alpha]$. Hence we have that

$$|\nabla^l \mathrm{Rm}|_{g(t)}(p) = |\nabla^l \mathrm{Rm}|_{\frac{1}{K}\tilde{g}(Kt)}(p) = K^{1+\frac{l}{2}} |\nabla^l \mathrm{Rm}|_{\tilde{g}(Kt)}(p) \leq K^{1+\frac{l}{2}} \frac{C}{(Kt)^{\frac{l}{2}}} = \frac{CK}{t^{\frac{l}{2}}}$$

for all $t \in \left(0, \frac{\alpha}{K}\right]$. Thus, as long as $K \geq 1$, at the central point p the constant $C(\alpha, r, K, l, n) > 0$ arising in Theorem 2.5.3 can be written in the form $C(\alpha, r, l, n)K$. This observation allows us to prove the following useful variant, which is implicit throughout Section 5 of [ST17].

Lemma 2.5.4 (Local Shi decay; Lemma B.1 in [MT18]). *Let $(\mathcal{M}^n, g(t))$ be a smooth Ricci flow for $t \in [0, T]$, and assume for some $R > 0$ and $x_0 \in \mathcal{M}$ that $\mathbb{B}_{g(0)}(x_0, R) \subset\subset \mathcal{M}$. Moreover, suppose that for all $0 < t \leq T$ we have $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ throughout $\mathbb{B}_{g(0)}(x_0, R)$ for some $c_0 > 0$. Then for any $\varepsilon \in (0, R)$, there exists $\hat{T} = \hat{T}(c_0, n, \varepsilon) > 0$ and, for $l \in \mathbb{N}$, there exists $C_l = C_l(l, c_0, n, \varepsilon) > 0$ such that if $0 < \tau \leq \min\{T, \hat{T}\}$ then we have $\mathbb{B}_{g(\tau)}(x_0, R - \varepsilon) \subset \mathbb{B}_{g(0)}(x_0, R)$ and*

$$|\nabla^l \mathrm{Rm}|_{g(t)} \leq \frac{C_l}{t^{1+\frac{l}{2}}} \tag{2.5.3}$$

throughout $\mathbb{B}_{g(\tau)}(x_0, R - \varepsilon) \times (0, \tau]$.

Proof of Lemma 2.5.4. Let $\beta = \beta(n) \geq 1$ be the constant arising in the shrinking balls lemma 2.4.6. Define $\hat{T} := \min\left\{c_0, \frac{\varepsilon^2}{9\beta^2 c_0}\right\} > 0$ and let $0 < \tau \leq \min\{T, \hat{T}\}$. The c_0/t curvature bound means that from Lemma 2.4.6 we deduce that $\mathbb{B}_{g(\tau)}(x_0, R - \varepsilon) \subset \mathbb{B}_{g(0)}\left(x_0, R - \frac{2\varepsilon}{3}\right) \subset \mathbb{B}_{g(0)}(x_0, R)$.

Let $x \in \mathbb{B}_{g(\tau)}(x_0, R - \varepsilon)$, $t \in (0, \tau]$, and consider $\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x, \frac{\varepsilon}{3}\right)$. Then, as we have just shown, we have $x \in \mathbb{B}_{g(0)}\left(x_0, R - \frac{2\varepsilon}{3}\right)$, hence via the shrinking balls lemma 2.4.6 we have $\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x, \frac{\varepsilon}{3}\right) \subset \mathbb{B}_{g(0)}(x, \frac{2\varepsilon}{3}) \subset \mathbb{B}_{g(0)}(x_0, R) \subset\subset M$.

Thus $\overline{\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x, \frac{\varepsilon}{4}\right)} \subset\subset \mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x, \frac{\varepsilon}{3}\right)$ and $|\mathrm{Rm}|_{g(s)} \leq \frac{2c_0}{t}$ throughout $\mathbb{B}_{g\left(\frac{t}{2}\right)}\left(x, \frac{\varepsilon}{3}\right)$ for all $s \in \left[\frac{t}{2}, t\right]$. We can apply Theorem 2.5.3 to the Ricci flow $s \mapsto g(s + t/2)$ for $s \in [0, t/2]$, with $r := \frac{\varepsilon}{4}$, $K := \frac{2c_0}{t} \geq 1$ and $\alpha := c_0$ to deduce that for a constant $C = C\left(l, c_0, n, \varepsilon\right) > 0$ we have

$$\left|\nabla^l \mathrm{Rm}\right|_{g\left(s+\frac{t}{2}\right)}(x) \leq \frac{2c_0 C}{s^{\frac{l}{2}} t} \tag{2.5.4}$$

for all $s \in \left(0, \frac{t}{2}\right]$. Here we have used our prior observation that if $K \geq 1$ then, at the central point x, the constant $C\left(\alpha, K, r, m, n\right)$ arising in Theorem 2.5.3 can be written in the form $C\left(\alpha, r, m, n\right) K$. Restricting to $s = t/2$ then gives (2.5.3) as required. ∎

By appealing to this localised version of Shi's estimates it is possible to add curvature derivative estimates, for all orders, to the conclusions of Theorem 2.4.12. The precise form of such derivative estimates is included in the full statement of the *Mollification theorem* of Simon and Topping in Theorem 1.1 of [ST17].

In Section 4.4 we need Lemma 8.1 in [ST17], which is itself a special case of a result of B.L. Chen's Theorem 3.1 in [Chen09]. For convenience, we state the result below.

Lemma 2.5.5 (Variant of Theorem 3.1 in [Chen09]; Lemma 8.1 in [ST17])**.** *Suppose that $\mathcal{M}$ is a smooth n-dimensional manifold and $g(t)$ a smooth Ricci flow solution, defined throughout $\mathcal{M} \times [0, T]$, such that for some $x \in \mathcal{M}$ and $r > 0$, and all $t \in [0, T]$, we have $\mathbb{B}_{g(t)}(x, r) \subset\subset \mathcal{M}$. Further assume that for all $0 < t \leq T$ we have $|\mathrm{Rm}|_{g(t)} \leq \frac{c_0}{t}$ throughout $\mathbb{B}_{g(t)}(x, r)$, for some $c_0 \geq 1$. Then if $|\mathrm{Rm}|_{g(0)} \leq r^{-2}$ on $\mathbb{B}_{g(0)}(x, r)$, we must have, for all $0 \leq t \leq T$, that*

$$|\mathrm{Rm}|_{g(t)}(x) \leq e^{C(n)c_0} r^{-2}. \tag{2.5.5}$$

We will also require Lemma 8.2 in [ST17] in Section 4.4; again for convenience we state the precise result below.

Lemma 2.5.6 (Lemma 8.2 in [ST17])**.** *Suppose that $\mathcal{M}$ is a smooth n-dimensional manifold and $g(t)$ a smooth Ricci flow solution, defined throughout $\mathcal{M} \times [0, T]$, with $\mathbb{B}_{g(0)}(x, r) \subset\subset \mathcal{M}$ for some $x \in \mathcal{M}$ and $r > 0$. Further suppose that $|\mathrm{Rm}|_{g(t)} \leq r^{-2}$ throughout $\mathbb{B}_{g(0)}(x, r) \times [0, T]$, and that, for some $l_0 \in \mathbb{N}$, we have that $|\nabla^l \mathrm{Rm}|_{g(0)} \leq r^{-2-l}$ throughout $\mathbb{B}_{g(0)}(x, r)$ for all $l \in \{1, \ldots, l_0\}$. Then there exists $C \in (0, \infty)$, depending only on l_0, n and an upper bound for T/r^2 such that for every $l \in \{1, \ldots, l_0\}$ and every $0 \leq t \leq T$, we have*

$$|\nabla^l \mathrm{Rm}|_{g(t)}(x) \leq C r^{-2-l}. \tag{2.5.6}$$

2.6. Pseudolocality

The Ricci flow equation may be viewed as a non-linear heat equation for the metric g; if one chooses harmonic coordinates, for example, then, for each pair i, j of indices, the components of Ric are given by

$$\mathrm{Ric}_{ij} = -\frac{1}{2}\Delta g_{ij} + \text{lower order terms.} \tag{2.6.1}$$

Such a viewpoint makes it tempting to expect that Ricci flows should exhibit the same properties as solutions of the standard linear heat equation. However, the pseudolocality theorem obtained by Perelman in the first of his seminal papers [Per02] establishes *improved* control for Ricci flow solutions that is simply not true for solutions to the standard linear heat equation.

Roughly speaking this theorem asserts that if a region is initially well-controlled (in some sense) then it cannot suddenly look highly non-trivial. The result effectively tells us that the Ricci flow is principally local; whilst the speed of propagation is infinite, there is a definite delay before regions of large curvature can significantly affect regions of controlled curvature. Control of this form is simply not true for solutions to the standard linear heat equation, and it is the inherent nonlinearity in the Ricci flow equation that gives rise to this advantageous damping affect.

The precise result obtained by Perelman is the following theorem.

Theorem 2.6.1 (Pseudolocality, Theorem 10.1 in [Per02])**.** *For any $\alpha > 0$ there exists $\varepsilon, \delta > 0$ with the following property. Suppose $0 < r_0 < \infty$ and $(\mathcal{M}^n, g(t))$ is a smooth complete n-dimensional Ricci flow for $0 \leq t \leq (\varepsilon r_0)^2$ of bounded curvature. Assume for some fixed $x_0 \in \mathcal{M}$ that $\mathrm{R}_{g(0)} \geq -r_0^{-2}$ throughout $\mathbb{B}_{g(0)}(x_0, r_0)$, and for every $\Omega \subset \mathbb{B}_{g(0)}(x_0, r_0)$ we have $\mathrm{Vol}(\partial\Omega)^n \geq (1-\delta)n^n\omega_n \mathrm{Vol}\,(\Omega)^{n-1}$ where ω_n is the volume of the Euclidean unit ball in $\mathbb{R}^n$. Then whenever $0 < t \leq (\varepsilon r_0)^2$ and $d_{g(t)}(x_0, x) < \varepsilon r_0$ we have $|\mathrm{Rm}|_{g(t)}(x) \leq \alpha t^{-1} + (\varepsilon r_0)^{-2}$.*

The control required by these two hypotheses is frequently referred to as "almost Euclidean", with the isoperimetric hypothesis being termed an "almost Euclidean isoperimetric inequality". Together the scalar curvature lower bound and the almost Euclidean isoperimetric inequality ensure that the curvature of the initial region cannot be too far from 0, justifying the "almost Euclidean terminology." In some sense the scalar curvature lower bound prevents too much negative curvature (recalling that a manifold with constant negative curvature will automatically satisfy the almost Euclidean isoperimetric hypothesis), whilst the almost Euclidean isoperimetric hypothesis prevents too much positive curvature (a manifold of very large constant positive curvature will not satisfy this requirement).

Requiring $g(t)$ to be a complete flow is necessary, as can be seen via the following example provided by Peter Topping. We summarise the presentation of this example provided in Theorem

A.3 in [GT13]. The idea is to take $\mathcal{M} = S^2$ and equip it with an initial metric g_0 so that $(\mathcal{M}, g_0)$ arises from taking a cylinder of length 2 and radius r, with the ends capped off with hemispheres. A computation gives that $\mathrm{Vol}_{g_0}(\mathcal{M}) = 4\pi(r + r^2)$, and hence the two-dimensional existence theory, Theorem 2.3.2, gives a smooth Ricci flow solution $g(t)$, starting at $g(0) = g_0$, and existing until time $T := \frac{1}{8\pi}\mathrm{Vol}_{g_0}(\mathcal{M}) = \frac{1}{2}(r + r^2)$, with $\inf\left\{\mathrm{K}_{g(t)}(x) : x \in \mathcal{M}\right\} \to \infty$ as $t \uparrow T$.

Taking x_0 midway along the cylindrical part of $\mathcal{M}$, we see that $g(0)$ is flat on $\overline{\mathbb{B}_{g_0}(x_0, 1)}$, and restricting the exponential map $\exp_{x_0}$ to the closed unit 2-disc $\mathcal{D}$ in the tangent space $T_{x_0}\mathcal{M}$, so that the image avoids the hemisphere caps, we may pull the flow back to $\mathcal{D}$. The resulting Ricci flow is initially the flat unit 2-disc, with the curvature blowing up everywhere by time T. By choosing $r > 0$ sufficiently small, the curvature can be made to blow up everywhere arbitrarily quickly, which in turn provides a counterexample to pseudolocality when the flow $g(t)$ is allowed to be incomplete.

There are numerous conditions which could be called "almost Euclidean". Indeed Perelman proves another variant of pseudolocality in [Per02] in which both the hypotheses and conclusion are stronger than those of Theorem 10.1 in [Per02], i.e. Theorem 2.6.1 above. The precise result is the following.

Theorem 2.6.2 (Theorem 10.3 in [Per02])**.** *There exists $\varepsilon, \delta > 0$ for which the following is true. Suppose $r_0 \in (0, \infty)$ and $(\mathcal{M}^n, g(t))$ is a smooth complete n-dimensional Ricci flow, defined for all times $0 \leq t \leq (\varepsilon r_0)^2$, and having bounded curvature. Suppose that for a fixed $x_0 \in \mathcal{M}$ we have $|\mathrm{Rm}|_{g(0)} \leq r_0^{-2}$ throughout $\mathbb{B}_{g(0)}(x_0, r_0)$ and $\mathrm{Vol}\mathbb{B}_{g(0)}(x_0, r_0) \geq (1-\delta) r_0^n \omega_n$. Then we have $|\mathrm{Rm}|_{g(t)}(x) \leq (\varepsilon r_0)^{-2}$ whenever $0 \leq t \leq (\varepsilon r_0)^2$ and $d_{g(t)}(x_0, x) < \varepsilon r_0$.*

The conclusion illustrates that the initial time curvature bound propagates forward for a definite amount of time. A result of Chen in [Chen09] provides a similar example of the same phenomenon in two dimensions under weaker hypotheses

Theorem 2.6.3 (Variant of Proposition 3.9 in [Chen09])**.** *Let $g(t)$ be a smooth Ricci flow on a smooth surface $\mathcal{M}^2$ defined for all $t \in [0, T]$. Let $x_0 \in \mathcal{M}$ and assume, for some $r_0 > 0$, that for all $t \in [0, T]$ we have $\mathbb{B}_{g(t)}(x_0, r_0) \subset\subset \mathcal{M}$. Given $v_0 > 0$, suppose that $|\mathrm{K}_{g(0)}| \leq r_0^{-2}$ throughout $\mathbb{B}_{g(0)}(x_0, r_0)$, and $\mathrm{Vol}\mathbb{B}_{g(0)}(x_0, r_0) \geq v_0 r_0^2$. Then there exists a constant $A = A(v_0) > 0$ such that*

$$\forall (x, t) \in \mathbb{B}_{g(t)}(x_0, r_0/2) \times \left[0, \min\left\{T, A r_0^2\right\}\right] \qquad \textit{we have} \qquad |\mathrm{K}_{g(t)}(x)| \leq 2 r_0^{-2}.$$

Completeness has been replaced by requiring the ball $\mathbb{B}_{g(t)}(x_0, r_0)$ to remain compactly contained in $\mathcal{M}$ throughout the flow. This condition is not satisfied by the flow constructed in the counterexample of Peter Topping, but is of course automatically satisfied by complete Ricci flows.

Moreover, the Ricci flow is no longer required to be of bounded curvature. This is, to our knowledge, the only pseudolocality result valid for flows with unbounded curvature, and in dimensions $n \geq 3$ the unbounded curvature case of pseudolocality remains an interesting open question.

A more recent example of the pseudolocality phenomenon under different hypotheses is Proposition 3.1 of [TW12]. It establishes the same curvature estimates achieved in Theorem 2.6.1 but now assuming an almost Euclidean lower Ricci bound, and an almost Euclidean lower volume bound for the initial ball. The precise result is the following theorem.

Theorem 2.6.4 (Variant of proposition 3.1 in [TW12])**.** *Let $n \in \mathbb{N}$ and $\alpha > 0$. Then there exist constants $\delta = \delta(n, \alpha) > 0$ and $\varepsilon = \varepsilon(n, \alpha) > 0$ for which the following is true. Suppose that $\mathcal{M}$ is a smooth n-dimensional manifold and that $g(t)$ is a smooth complete Ricci flow solution, defined throughout $\mathcal{M} \times [0, \varepsilon^2]$, and having bounded curvature. Assume that $x \in \mathcal{M}$ and that we have both that $\mathrm{Ric}_{g(0)} \geq -(n-1)\delta^4$ throughout $\mathbb{B}_{g(0)}\left(x, \delta^{-1}\right)$ and that $\mathrm{Vol}\mathbb{B}_{g(0)}\left(x, \delta^{-1}\right) \geq (1-\delta)\delta^{-n}\omega_n$. Then we have the curvature bound $|\mathrm{Rm}|_{g(t)}(z) \leq \frac{\alpha}{t} + \frac{1}{\varepsilon^2}$ whenever $t \in (0, \varepsilon^2]$ and $d_{g(t)}(x, z) < \varepsilon$.*

In the interest of completeness, it is worth remarking that recent work of Fabio Cavalletti and Andrea Mondino establishes that the conditions assumed in Proposition 3.1 in [TW12], i.e. Theorem 2.6.4 above, imply that the hypotheses of Theorem 10.1 in [Per02], i.e. Theorem 2.6.1 above, are satisfied on a strictly smaller initial region, see [CM17].

More recently, Miles Simon and Peter Topping obtain a pseudolocality-type result in dimension three valid outside the *almost Euclidean* setting. In particular, a consequence of Theorem 1.1 in [ST16] is that even when the hypotheses of Proposition 3.1 in [TW12] are not close to their Euclidean counterparts, one may still conclude C/t curvature decay for some $C > 0$.

2.7. Cheeger-Gromov Convergence

Sequences of Riemannian manifolds satisfying some form of curvature bounds arise frequently in geometric-analysis. A particularly powerful technique for studying singularities of geometric flows is to "blow up" around the singularity and to somehow pass to the limit, in which the underlying geometry of the singularity should be more easily accessible. They also arise during contradiction arguments, in which case not only is the existence of a limit important, but often the regularity possessed by the limit is essential.

In order to make sense of such limits we require a suitable notion of convergence for a sequence of Riemannian manifolds. Given the inherent invariance of Riemannian geometry under the action of diffeomorphisms, it seems reasonable to ask for a notion of convergence that is diffeomorphism invariant.

Definition 2.7.1 (Cheeger-Gromov convergence of manifolds)**.** A sequence $(\mathcal{M}_i, g_i, x_i)$ of smooth, pointed, complete n-dimensional Riemannian is said to *converge smoothly* to a smooth pointed n-dimensional Riemannian manifold $(\mathcal{N}, h, x_0)$ as $i \to \infty$ if there exists a sequence of domains $\Omega_i \subset\subset \mathcal{N}$, with $x_0 \in \Omega_i$ for every i, exhausting $\mathcal{N}$, and a sequence of smooth maps $\varphi_i : \Omega_i \to \mathcal{M}_i$, mapping x_0 to x_i, diffeomorphic onto their image and such that $\varphi_i^* g_i \to h$ smoothly locally on $\mathcal{N}$ as $i \to \infty$.

This notion is designed to deal with non-compact sequences and limits. It is worth remarking that if the limit $\mathcal{N}$ were compact, then we would necessarily have that $\Omega_i = \mathcal{N}$ for sufficiently large i. In this case the maps φ_i will be defined throughout $\mathcal{N}$, mapping the whole of $\mathcal{N}$ diffeomorphically to $\mathcal{M}_i$, for sufficiently large i. However, it is important to observe that we can have each $\mathcal{M}_i$ compact, but the limit $\mathcal{N}$ be non-compact. For example, consider the sequence of cylinders $S^1 \times [-i, i]$ with capped off unit hemispherical ends. Equipping each manifold with the obvious metric g_i, we see that this sequence converges to $S^1 \times \mathbb{R}$ in the Cheeger-Gromov sense.

Under this notion of convergence, complete limits are unique in the following sense. Suppose $(\mathcal{N}_1, h_1, x_1)$ and $(\mathcal{N}_2, h_2, x_2)$ are both smooth complete Cheeger-Gromov limits of the same sequence of pointed manifolds. Then there exists an isometry $I : (\mathcal{N}_1, h_1) \to (\mathcal{N}_2, h_2)$ mapping x_1 to x_2, see Lemma B.3 in [Top12] for example. In fact, as we will see in Lemma 3.3.1 in Chapter 3, Lemma B.3 in [Top12] carries over to a more general situation.

The inclusion of base points in the definition is necessary to give rise to well-defined limits. Consider, for example, a cylinder $S^1 \times [0, \infty)$ capped off with a unit hemispherical end. Call this manifold $\mathcal{M}$ and equip it with the obvious metric g. First consider the sequence of base points $p_i \in \mathcal{M}$ where p_i is chosen to be a point a distance i away from the join between the cylinder and the hemisphere. Then the sequence $(\mathcal{M}, g, p_i)$ Cheeger-Gromov converges to $S^1 \times \mathbb{R}$. However, if we consider the fixed base point $q \in \mathcal{M}$ as the tip of the hemisphere, then the sequence $(\mathcal{M}, g, q)$ Cheeger-Gromov converges to the original manifold $S^1 \times [0, \infty)$.

With this notion of convergence we may state the powerful Cheeger-Gromov compactness theorem. Whilst originally due to Gromov (see [Gro99], for example), various versions of this result have appeared in a number of works. See, for example, works of Katsuda [Kat85], Peters [Pet85], Greene and Wu [GW88], Fukaya [Fuk88], Kasue [Kas89] and Hamilton [Ham95].

Theorem 2.7.2 (Global Cheeger-Gromov compactness)**.** *Let $(\mathcal{M}_i, g_i, x_i)$ be a sequence of smooth, complete, pointed n-dimensional Riemannian manifolds. Assume that*

1. *for every $r > 0$ and every $l \in \mathbb{N} \cup \{0\}$ there exists $A = A(r, l) > 0$ such that for every $i \in \mathbb{N}$ we have $|\nabla^l \mathrm{Rm}|_{g_i} \leq A$ throughout $\mathbb{B}_{g_i}(x_i, r) \subset\subset \mathcal{M}_i$, and*

2. *there exists $B > 0$ such that for every $i \in \mathbb{N}$ we have $\mathrm{inj}_{g_i}(x_i) \geq B$.*

Then, after passing to a subsequence in i, there exists a smooth, complete, pointed n-dimensional Riemannian manifold $(\mathcal{N}, h, q)$ such that $(\mathcal{M}_i, g_i, x_i) \to (\mathcal{N}, h, q)$ in the smooth Cheeger-Gromov sense as $i \to \infty$.

Both hypotheses 1 and 2 in Theorem 2.7.2 are necessary. Without the curvature control of hypothesis 1 one could obtain convergence of smooth manifolds to a Euclidean cone. Without the injectivity radius control of hypothesis 2, we could consider the sequence $S^1(r_i) \times \mathbb{R}$ of cylinders of radius $r_i := \frac{1}{i}$, which degenerates to the one dimensional straight line $\mathbb{R}$ as $i \to \infty$.

Given the curvature bounds in hypothesis 1 of Theorem 2.7.2 the following result, which is Theorem 4.7 in [CGT82], tells us that a lower bound on a unit ball implies a lower injectivity radius bound at the central point.

Theorem 2.7.3 (Volume bound implies injectivity radius bound; Theorem 4.7 in [CGT82])**.** *Suppose $(\mathcal{M}, g)$ is a smooth n-dimensional Riemannian manifold and fix a point $x \in \mathcal{M}$. Assume $\mathbb{B}_g(x, 1) \subset\subset \mathcal{M}$, and that there exist constants $K, v > 0$ for which $|\mathrm{Rm}|_g \leq K$ throughout $\mathbb{B}_g(x, 1)$ and $\mathrm{Vol}\mathbb{B}_g(x, 1) \geq v$. Then there exists $I_0 = I_0(v, K, n) > 0$ such that $\mathrm{inj}_g(x) \geq I_0$.*

Evidently Theorem 2.7.3 allows us to replace hypothesis 2 of Theorem 2.7.2 with the requirement that there exists $v > 0$ such that for every $i \in \mathbb{N}$ we have $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v > 0$.

Hamilton's compactness theorem for Ricci flows, first appearing in [Ham95], may be deduced from the Cheeger-Gromov compactness theorem for Riemannian manifolds (Theorem 2.7.2). In order to do so we must first make sense of what it means for a sequence of flows to converge.

Definition 2.7.4 (Cheeger-Gromov convergence of Ricci flows)**.** For each $i \in \mathbb{N}$ let $\mathcal{M}_i$ be an n-dimensional smooth Riemannian manifold and $g_i(t)$ be a Ricci flow solution on $\mathcal{M}_i$, defined for all $t \in [a, b]$ for some $-\infty \leq a < b \leq +\infty$ independent of i. Let $x_i \in \mathcal{M}_i$ for each $i \in \mathbb{N}$. Finally let $\mathcal{N}$ be a smooth n-dimensional Riemannian manifold and $h(t)$ a smooth Ricci flow on $\mathcal{N}$ defined for all $t \in [a, b]$ and let $q \in \mathcal{N}$. We say that $(\mathcal{M}_i, g_i(t), x_i) \to (\mathcal{N}, h(t), q)$ in the smooth Cheeger-Gromov sense as $i \to \infty$ if there exist

1. a sequence of domains $\Omega_i \subset\subset \mathcal{N}$ exhausting $\mathcal{N}$ and all containing q, and
2. a sequence of smooth maps $\varphi_i : \Omega_i \to \mathcal{M}_i$, mapping q to x_i and diffeomorphic onto their image,

such that $\varphi_i^* g_i(t) \to h(t)$ smoothly locally on $\mathcal{N} \times [a, b]$ as $i \to \infty$.

In [Ham95], Hamilton shows how Shi's derivatives estimates (cf. Theorem 2.5.2) may be combined with the Cheeger-Gromov compactness theorem (Theorem 2.7.2) at a single time $t > 0$ to prove the following compactness theorem.

Theorem 2.7.5 (Hamilton's Ricci flow compactness theorem [Ham95])**.** *Let $(\mathcal{M}_i, g_i(t), x_i)$, for $i \in \mathbb{N}$, be a sequence of smooth, complete, pointed n-dimensional Ricci flows, defined for all $t \in [0,T]$ for some $T > 0$. Further suppose that there exist constants $A, B \in (0,\infty)$ and a time $t_0 \in (0,T)$ such that for every $i \in \mathbb{N}$ we have*

1. $|\mathrm{Rm}|_{g_i(t)} \leq A$ *throughout* $\mathcal{M}_i \times [0,T]$, *and*

2. $\mathrm{inj}_{g_i(t_0)}(x_i) \geq B$.

Then there exists a smooth n-dimensional manifold $\mathcal{N}$, a smooth complete Ricci flow $h(t)$ on $\mathcal{N}$, defined for all $t \in (0,T]$, and a point $q \in \mathcal{N}$ such that, after passing to a subsequence in i, we have that $(\mathcal{M}_i, g_i(t), x_i) \to (\mathcal{N}, h(t), q)$ as $i \to \infty$.

Convergence is not claimed, and indeed cannot be expected, at the time $t = 0$. This is a direct consequence Shi's derivative estimates (Theorem 2.5.2) only becoming valid after the parabolic nature of the flow has had some time in which to smooth out the metric.

Motivated by the weaker local curvature bounds required in Theorem 2.7.2, it is natural to wonder if Theorem 2.7.5 remains valid with hypothesis 1 weakened to "for every $r \in (0,\infty)$ there exists $A = A(r) \in (0,\infty)$ such that for every $i \in \mathbb{N}$ we have $|\mathrm{Rm}|_{g_i(t)} \leq A$ throughout $\mathbb{B}_{g_i}(x_i, r) \times [0,T]$". Indeed, the local version of Shi's derivative estimates (Theorem 2.5.3) would still provide i independent bounds on the derivatives of $\mathrm{Rm}_{g_i(t)}$ for positive times $t > 0$. However, particular care is required with regards to the conclusions one wishes to establish. Asking for the limit Ricci flow to be complete is no longer reasonable, as seen by the counterexample provided by Topping in [Top11]. But, if we drop the requirement that the limit Ricci flow is complete, then Theorem 2.7.5 remains valid in this setting, see Theorem 1.2 in [Top11] for example.

There are many variants of Theorem 2.7.5 which we will not discuss here, some of which may be found in Appendix E of [KL06], for example. Our main concern will be with a local variant of Theorem 2.7.5, see Theorem 3.6.1, which is already implicit in [ST17], and will be the main focus of Chapter 3.

2.8. Gromov-Hausdorff Convergence

The Cheeger-Gromov notion of convergence is often too restrictive in naturally arising situations. For example, the setting of Ricci lower bounds is the natural framework for comparison geometry (the Bishop-Gromov volume comparison theorem, Theorem 2.2.1, being a prime example), hence studying sequences of manifolds satisfying uniform lower Ricci bounds is a common occurrence. However, only having lower Ricci bounds gives no guarantee that the full curvature bounds required by Cheeger-Gromov convergence will be satisfied. Instead we require a weaker, less restrictive notion of convergence to study such sequences.

The natural candidate for such a notion of convergence is *Gromov-Hausdorff convergence*, which gives a notion of convergence for sequences of metric spaces.

Definition 2.8.1 (Gromov-Hausdorff convergence; see [BBI01] or [Che01], for example)**.** Let (X_i, d_i), for $i \in \mathbb{N}$, be a sequence of compact metric spaces and (X, d_0) a compact metric space. Then we say $(X_i, d_i) \to (X, d_0)$ in the *Gromov-Hausdorff* sense as $i \to \infty$ if there exists a sequence of maps $f_i : X \to X_i$ such that given any $\varepsilon > 0$, there exists an $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have that

1. $|d_0(x, y) - d_i(f_i(x), f_i(y))| < \varepsilon$ for every $x, y \in X$, and
2. $X_i = (f_i(X))_\varepsilon := \{z \in X_i : \exists x \in X \text{ s.t. } d_i(f_i(x), z) < \varepsilon\}$.

We call a map satisfying both 1 and 2 an ε*-Gromov-Hausdorff approximation*, sometimes written ε-GH-approx.

Condition 1 in definition 2.8.1 imposes that the map is an ε*-almost isometry,* whilst condition 2 imposes that the map is ε*-almost surjective.* We can use the notion of ε-Gromov-Hausdorff approximations to define a notion of distance between two metric spaces. Given two compact metric spaces (X, d) and (Y, ρ), we define

$$d_{GH}(X, Y) := \inf\{\varepsilon > 0 : \exists \varepsilon - \text{GH-approxs } f : (X, d) \to (Y, \rho) \text{ and } h : (Y, \rho) \to (X, d)\}. \tag{2.8.1}$$

Then we have that the convergence $(X_i, d_i) \to (X, d)$ in the Gromov-Hausdorff sense as $i \to \infty$ is equivalent to $d_{GH}(X_i, X) \to 0$ as $i \to \infty$.

We will be interested in non-compact spaces and so we require the notion of *pointed Gromov-Hausdorff* convergence.

Definition 2.8.2. Let (X_i, d_i, x_i), for $i \in \mathbb{N}$, be a sequence of, possibly non-compact, pointed metric spaces and (X, d_0, x_0) a pointed metric space. Then we say $(X_i, d_i, x_i) \to (X, d_0, x_0)$ in the *pointed Gromov-Hausdorff* sense as $i \to \infty$ if, given any $r > 0$ and $\varepsilon > 0$, there exists an $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we can find maps $f_i^r : \mathbb{B}_{d_0}(x_0, r) \to \mathbb{B}_{d_i}(x_i, r)$ satisfying

1. $f_i^r(x_0) = x_i$,
2. $|d_0(x, y) - d_i(f_i^r(x), f_i^r(y))| < \varepsilon$ for every $x, y \in \mathbb{B}_{d_0}(x_0, r)$, and
3. $\mathbb{B}_{d_i}(x_i, r) \subset (f_i^r(\mathbb{B}_{d_0}(x_0, r)))_\varepsilon := \{z \in X_i : \exists x \in \mathbb{B}_{d_0}(x_0, r) \text{ s.t. } d_i(f_i^r(x), z) < \varepsilon\}$.

That is, the maps f_i^r for $i \geq i_0$ are ε*-Gromov-Hausdorff approximations* $\mathbb{B}_{d_0}(x_0, r)$ to $\mathbb{B}_{d_i}(x_i, r)$.

Considering the sequence of smooth manifolds $S^1(r_i) \times \mathbb{R}$ with $r_i := \frac{1}{i}$ gives an example of a sequence for which there is a limit in the pointed Gromov-Hausdorff sense, but for which there is no limit in the smooth Cheeger-Gromov sense. Thus, as expected, the pointed Gromov-Hausdorff notion of convergence is weaker than the smooth Cheeger-Gromov notion.

A powerful property of the pointed Gromov-Hausdorff convergence is *Gromov's compactness theorem*. Originally appearing in [Gro81], the particular variant below may be found as Theorem 5.3 in [Gro99].

Theorem 2.8.3 (Gromov's compactness theorem; Theorem 5.3 in [Gro99])**.** *Suppose* $(\mathcal{M}_i, g_i, x_i)$, *for* $i \in \mathbb{N}$, *is a sequence of smooth, complete, pointed* n*-dimensional Riemannian manifolds such that for every* $i \in \mathbb{N}$ *we have* $\mathrm{Ric}_{g_i} \geq -\alpha_0$, *throughout* $\mathcal{M}_i$, *for some* $\alpha_0 > 0$. *Then there exists a complete, locally compact, pointed metric space* (X, d, x_0) *such that, after passing to a subsequence in* i, *we have* $(\mathcal{M}_i, d_{g_i}, x_i) \to (X, d, x_0)$ *in the pointed Gromov-Hausdorff sense as* $i \to \infty$.

2.9. Ricci Limit Spaces

Given a sequence of smooth, complete, pointed n-dimensional Riemannian manifolds $(\mathcal{M}_i, g_i, x_i)$ satisfying, for some $\alpha_0 > 0$, that $\mathrm{Ric}_{g_i} \geq -\alpha_0$, we may appeal to Gromov's compactness theorem 2.8.3 to obtain a locally compact, complete pointed metric space (X, d, x_0) such that, after passing to a subsequence in i, we have that $(\mathcal{M}_i, d_{g_i}, x_i) \to (X, d, x_0)$ in the pointed Gromov-Hausdorff sense as $i \to \infty$. We refer to such metric spaces (X, d) as *Ricci limit spaces*. We expect, in some sense, the metric space (X, d) to have better regularity than that of an arbitrary metric space.

In order to study the regularity properties of (X, d), we need to be able to study the local properties at each point $x \in X$. To do so, we need to introduce the notion of *tangent cones*. Given a point $z \in X$ and a positive null sequence r_i for $i \in \mathbb{N}$, we may consider the sequence $(X, r_i^{-1}d, z)$ of pointed metric spaces. It can be shown (see Chapter 10 in [Che01] for example) that, after passing to a subsequence in i, there exists a complete metric space $(X_z, d_\infty, z_\infty)$ such that $(X, r_i^{-1}d, z) \to (X_z, d_\infty, z_\infty)$ in the pointed Gromov-Hausdorff sense as $i \to \infty$. Such limits $(X_z, d_\infty, z_\infty)$ are called *tangent cones at* z. Whilst there is no guarantee that there is a unique tangent cone at $z \in X$, they nevertheless provide a means of studying the local geometry of the limit space X at z.

The *regular set* of X, denoted $\mathcal{R}$, is defined to by

$$\mathcal{R} := \{z \in X : \text{Every tangent cone at } z \text{ is isometric to } \mathbb{R}^k \text{ for some } k \in \mathbb{N}_0\}. \tag{2.9.1}$$

It is important to digest that being in the regular set $\mathcal{R}$ is a purely pointwise notion. In particular, $x \in \mathcal{R}$ does not imply that there is some open neighbourhood of x contained in $\mathcal{R}$. The *singular* *set* of X is defined to be the set of all points not in the regular set, i.e. defined by

$$\mathcal{S} := X \setminus \mathcal{R}. \tag{2.9.2}$$

Given $\varepsilon > 0$, the *ε-regular set, $\mathcal{R}_\varepsilon$*, consists of all points $x \in X$ that are, in a scaled sense, within ε of being Euclidean on sufficiently small scales. To be precise, for each $k \in \mathbb{N}$ we define

$$\mathcal{R}^k_\varepsilon := \{x \in X : \exists r_0 > 0 \text{ s.t. for every } r \in (0, r_0) \text{ we have } d_{GH}\left(\mathbb{B}_d(x,r), \mathbb{B}^k(0,r)\right) < \varepsilon r\} \tag{2.9.3}$$

where $\mathbb{B}^k(0,r) := \{x \in \mathbb{R}^k : |x| < r\}$. Then we can define $\mathcal{R}_\varepsilon := \bigcup_{k\in\mathbb{N}} \mathcal{R}^k_\varepsilon$. Evidently we have $\mathcal{R} = \cap_{\varepsilon>0}\mathcal{R}_\varepsilon$, but it is worth noting that it is possible to have $\mathcal{R}_\varepsilon \cap \mathcal{S} \neq \emptyset$.

Before covering the regularity results obtained by Cheeger-Colding in the 1990s, it is instructive to consider what is reasonable to expect for the limit space (X, d). By returning to our previous example of the sequence of cylinders $S^1(r_i) \times \mathbb{R}$, for a positive null sequence r_i, we see that in general it is not reasonable to ask that $\dim_{\mathscr{H}}(X) = n$. Indeed, to rule out loss of dimension in the limit, we must impose a *noncollapsed* assumption.

Returning to our sequence $(\mathcal{M}^n_i, g_i, x_i)$ of smooth Riemmanian n-manifolds, we say that the Ricci limit space X is *collapsed* if $\liminf_{i\to\infty} \mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) = 0$, and we say it is *noncollapsed* if $\liminf_{i\to\infty} \mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) > 0$, which is, after passing to a subsequence in i, equivalent to having $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v > 0$ for all $i \in \mathbb{N}$. This situation is sometimes referred to as *weakly noncollapsed* as we are only imposing a uniform lower volume bound on a *single* unit ball, rather than on *every* unit ball. By appealing to Bishop-Gromov, Lemma 2.2.3, these two notions are seen to be equivalent in the case of uniformly bounded diameter. From now on we assume we are in the noncollapsed situation.

It is natural to wonder if it is possible to prove that $\mathcal{S} = \emptyset$. However, the following example illustrates that this expectation is not reasonable. Consider the standard two-sphere S^2 equipped with the usual round metric. Let $x \in S^2$ and cut-out a small disc, of radius $r << 1$, say. Then we can glue on a smoothed out Euclidean cone (of radius r) such that it meets S^2 tangentially. This ensures that after being equipped with the obvious metric, the Ricci curvature remains non-negative. Taking the limit in which the smoothed out cones converge to the non-smooth Euclidean cone we see that the limit of the sequence of points at the tip of the smoothed out cone must lie in $\mathcal{S}$. Moreover, a more elaborate construction along these lines, utilising the graph of $\sum_{i=1}^{\infty} 2^{-i}|x|$ over a dense set of rational points, we observe that we can even have that $\mathcal{S}$ is dense in X.

Nevertheless, Cheeger and Colding established a number of regularity results in the 1990s. We summarise a selection of these results for the noncollapsed setting, all of which may be found in [Che01], say.

Theorem 2.9.1 (Cheeger-Colding regularity [Che01])**.** *Let (X, d, x_0) be a Ricci limit space arising from a sequence $(\mathcal{M}_i^n, g_i, x_i)$, for $i \in \mathbb{N}$, of complete, smooth, pointed Riemannian n-manifolds satisfying, for given $\alpha_0, v_0 > 0$, that for every $i \in \mathbb{N}$ we have that $\mathrm{Ric}_{g_i} \geq -\alpha_0$ throughout $\mathcal{M}_i$ and $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$. Then*

1. $\dim_{\mathscr{H}}(X) = n$,
2. *the regular set $\mathcal{R}$ is connected, dense in X and for every $x \in \mathcal{R}$ we have that every tangent cone at x is isometric to $\mathbb{R}^n$,*
3. $\dim_{\mathscr{H}}(\mathcal{S}) \leq n - 2$, *and*
4. *for sufficiently small $\varepsilon > 0$ the interior of $\mathcal{R}_\varepsilon$, denoted $\mathcal{R}_\varepsilon^\circ$, is bi-Hölder homeomorphic to a smooth manifold.*

Conclusion 4 is particularly strong thanks to Theorem 5.14 in [CC97], which itself establishes that for every $\varepsilon > 0$ there exists $\delta \in (0, \varepsilon)$ for which $\mathcal{R}_\delta \subset \mathcal{R}_\varepsilon^\circ$. Hence, if the singular set $\mathcal{S} \neq \emptyset$, then $\mathcal{R}_\varepsilon^\circ$ will contain some points that are within the singular set. However, conclusion 4 tells us that these points are not badly singular. This naturally leads one to wonder if the entire limit space X, including all singular points in $\mathcal{S}$, is a smooth manifold throughout the entire limit space.

In dimension 4 and higher, this can be seen to be false. This is a result of the so-called Eguchi-Hanson metric constructed by Eguchi and Hanson in [EH78]. They construct a Ricci flat metric g on the tangent bundle TS^2 of the sphere S^2, which is rapidly asymptotic to the standard canonical flat metric outside the unit sphere bundle of TS^2. By considering $\varepsilon_i^2 g$ for a null sequence $\varepsilon_i > 0$, it can be shown that $(TS^2, \varepsilon_i^2 g, 0)$ converges, in the pointed Gromov-Hausdorff sense, to the quotient of $\mathbb{R}^4$ by the antipodal map $x \mapsto -x$, which is not homeomorphic to any topological manifold. Taking products with $\mathbb{R}^{n-4}$ provides counterexamples for every dimension $n \geq 4$.

However, in [ST17], Miles Simon and Peter Topping establish that, in dimension 3, noncollapsed Ricci limit spaces are globally homeomorphic to topological manifolds.

Theorem 2.9.2 (Three-dimensional Ricci limit spaces are topological manifolds; Corollary 1.5 in [ST17])**.** *Suppose that $(\mathcal{M}_i^3, g_i, x_i)$, for $i \in \mathbb{N}$, is a sequence of smooth, complete, pointed three-dimensional Riemannian manifolds satisfying, for given $\alpha_0, v_0 > 0$, that $\mathrm{Ric}_{g_i} \geq -\alpha_0$ throughout $\mathcal{M}_i$ and $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$ for every $i \in \mathbb{N}$.*

Then there exists a topological three-manifold M, a distance metric $d : M \times M \to [0, \infty)$, generating the same topology as M and making (M, d) a complete metric space, such that, after passing to a subsequence in i, we have $(\mathcal{M}_i, d_{g_i}, x_i) \to (M, d, x_0)$ in the pointed Gromov-Hausdorff sense, for some $x_0 \in M$, as $i \to \infty$. Moreover, the charts for M may be taken to be bi-Hölder with respect to d.

In fact, in Theorem 1.4 in [ST17], Simon and Topping establish that given any point $x \in X$, including any singular point, there is a neighbourhood of x that is bi-Hölder homeomorphic to a ball in $\mathbb{R}^3$. Central to Simon and Topping's proof of Theorem 2.9.2 above is their use of Ricci flow to locally *'mollify'* the Riemannian manifolds $(\mathcal{M}_i, g_i)$ via their *'mollification theorem'*, Theorem 2.4.12, in the spirit of early work of Simon e.g. [Sim02, Sim12].

Chapter 3

Local Compactness Theorem

3.1. Outline of Chapter

The goal of this chapter is to prove a Cheeger-Gromov-Hamilton type compactness theorem for local Ricci flows, see Theorem 3.6.1, which is already implicit in [ST17]. Before obtaining our desired compactness result for local Ricci flows, we must first establish the corresponding local Cheeger-Gromov compactness theorem, see Theorem 3.2.1.

We wish to establish compactness with only local curvature and volume estimates on a ball of radius R, without requiring completeness of our sequence. Our aim is to obtain a sequence of diffeomorphisms, defined on an exhaustion of this ball of radius R, such that the sequence of pull backs converges to a smooth Riemannian metric. Our strategy will be to prove compactness for a fixed radius $r \in (0, R)$, see Lemma 3.4.1, before using this result on a sequence of radii $r_i \uparrow R$ as $i \to \infty$ in order to establish Theorem 3.2.1.

Implementing this strategy will require being able to relate limits arising from different radii. This will be possible thanks to the uniqueness statement for Cheeger-Gromov limits, which we explicitly provide in Lemma 3.3.1. After obtaining a countable collection of limits via an appropriate diagonal subsequence we will need to construct a single manifold on which our final limit metric will live. Details of how this may be done are provided in our *smooth manifold construction theorem* (Theorem 3.3.2). Moreover, several statements regarding where the convergence is valid will require careful understanding of how distance functions behave under local smooth Cheeger-Gromov convergence; thus we record some key observations about this in Lemma 3.3.3, which is Lemma 6.1 in [ST17].

Finally, with Theorem 3.2.1 established, we prove our local Ricci flow compactness theorem, Theorem 3.6.1. For this we additionally require some supplementary lemmata of Simon and Topping regarding local Ricci flow; all of which may be found in Section 2.4.

The remainder of this chapter is set out as follows. In Section 3.2 we provide a precise statement of the localised Cheeger-Gromov compactness theorem that we need to prove. In Section 3.3 we present a uniqueness of Cheeger-Gromov limits statement, valid even in the setting of incomplete limits. Within this section, we present results concerning the behaviour of distance functions under local smooth Cheeger-Gromov convergence (Lemma 3.3.3) and how a single limit manifold can be extracted from a countable collection corresponding to different radii (Theorem 3.3.2). In Section 3.4 we establish Theorem 3.2.1 under the assumption that Lemma 3.4.1, itself claiming compactness at a fixed radius $r \in (0, R)$, is valid. In Section 3.5 we prove Lemma 3.4.1. Finally, in Section 3.6 we prove Theorem 3.6.1.

3.2. Statement of Local Compactness Theorem

The following theorem localises the well-known Cheeger-Gromov compactness theorem, i.e. Theorem 2.7.2.

Theorem 3.2.1 (Local compactness; Lemma B.2 in [MT18]). *Suppose $(\mathcal{M}_i^n, g_i)$ is a sequence of smooth n-dimensional Riemannian manifolds, not necessarily complete, and that $x_i \in \mathcal{M}_i$ for each i. Suppose that, for some $R > 0$, we have $\mathbb{B}_{g_i}(x_i, R) \subset\subset \mathcal{M}_i$ for each i, that $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, R) \geq v > 0$ and that (for each l) we have $|\nabla^l \mathrm{Rm}|_{g_i} \leq C_l$ throughout $\mathbb{B}_{g_i}(x_i, R)$, for constants C_l and v that are independent of i (with C_l allowed to depend on l).*

Then after passing to an appropriate subsequence in i, there exist a smooth, typically incomplete n-dimensional Riemannian manifold $(\mathcal{N}, g_\infty)$, a point $x_0 \in \mathcal{N}$ with $\mathbb{B}_{g_\infty}(x_0, r) \subset\subset \mathcal{N}$ for every $r \in (0, R)$, and a sequence of smooth maps $\varphi_i : \mathbb{B}_{g_\infty}(x_0, \frac{i}{i+1}R) \to \mathcal{M}_i$, diffeomorphic onto their images and mapping x_0 to x_i, such that $\varphi_i^* g_i \to g_\infty$ smoothly locally on $\mathbb{B}_{g_\infty}(x_0, R)$.

Remark 3.2.2. It is not reasonable to ask that $\mathbb{B}_{g_\infty}(x_0, R) \subset\subset \mathcal{N}$. Indeed, taking $(\mathcal{M}_i, g_i)$ to be the flat disc of radius $R + \frac{1}{i}$ we want the limit $(\mathcal{N}, g_\infty)$ to be the flat disc of radius R.

3.3. Uniqueness of Cheeger-Gromov Limits

The following result records the sense in which Cheeger-Gromov limits are unique. It provides an explicit extension of Lemma B.3 in [Top12] to the incomplete setting, though it is well-known that this is possible.

Lemma 3.3.1 (Uniqueness of limits). *Suppose $(\mathcal{M}_i^n, g_i)$ is a sequence of smooth, not necessarily complete, n-dimensional Riemannian manifolds. Assume we have smooth (possibly incomplete) n-dimensional Riemannian manifolds $(\mathcal{N}_1^n, h_1)$ and $(\mathcal{N}_2^n, h_2)$ with points $x_0 \in \mathcal{N}_1$ and $y_0 \in \mathcal{N}_2$ and connected open neighbourhoods $O_1 \subset\subset \mathcal{N}_1$ of x_0 and $O_2 \subset\subset \mathcal{N}_2$ of y_0. Further suppose* *there are sequences of smooth maps $\varphi_i : O_1 \to \mathcal{M}_i$ and $\omega_i : O_2 \to \mathcal{M}_i$, diffeomorphic onto their images, with $\varphi_i(x_0) = \omega_i(y_0)$ and $\varphi_i(O_1) \subset \omega_i(O_2) \subset\subset \mathcal{M}_i$ for all $i \in \mathbb{N}$, and such that $\varphi_i^* g_i \to h_1$ smoothly uniformly on O_1 and $\omega_i^* g_i \to h_2$ smoothly uniformly on O_2. Then there exists a smooth map $I : O_1 \to O_2$ that is an isometry when domain and target are given the metrics h_1 and h_2 respectively, and which sends x_0 to y_0.*

To clarify, by isometry we mean that the metrics $(I^{-1})^* h_1$ and h_2 coincide where both are defined. In particular, there is no claim that distances are preserved, and it is not reasonable to ask for such a conclusion.

Proof of Lemma 3.3.1. The details of the proof of Lemma B.3 in [Top12] carry across verbatim. For completeness we provide a brief summary of the argument.

Consider the smooth maps $J_i : O_1 \to O_2$ for $i \in \mathbb{N}$ given by $J_i := \omega_i^{-1} \circ \varphi_i$ and satisfying that $J_i(x_0) = y_0$ for all $i \in \mathbb{N}$. These maps yield a sequence of linear maps $(J_i)_* : T_{x_0}\mathcal{N}_1 \to T_{y_0}\mathcal{N}_2$. After potentially passing to a subsequence in i, we may assume the sequence $(J_i)_*$ converges smoothly to a limit map $J_* : T_{x_0}\mathcal{N}_1 \to T_{y_0}\mathcal{N}_2$ identifying the respective tangent spaces.

The smooth convergence of the hypotheses then tell us that $J_i^* h_2 \to h_1$ smoothly uniformly on O_1. Therefore, on some open neighbourhood of $0 \in T_{y_0}\mathcal{N}_2$ we have $\exp_{J_i(x_0),(J_i)_* h_1} \to \exp_{y_0,h_2}$, and so, by writing $J_i = \exp_{J_i(x_0),(J_i)_* h_1} \circ (J_i)_* \circ \exp^{-1}_{x_0,h_1}$ near x_0, we find that $J_i \to \exp_{y_0,h_2} \circ J_* \circ \exp^{-1}_{x_0,h_1}$ smoothly near x_0, and this limit is then necessarily an isometry.

Hence $x_0 \in A := \{z \in O_1 : J_i$ converges smoothly to an isometry in a neighbourhood of $z\}$ where the isometry is with respect to the metrics h_1 on O_1 and h_2 on O_2.

If $p \in A$ then a minor alteration to the above argument gives that any geodesic ball centred at p, which is compactly contained in O_1 and with radius smaller than the injectivity radius at p, lies within A. Combined with connectedness this yields that $A = O_1$ and thus J_i converges smoothly to a local isometry $I : (O_1, h_1) \to (O_2, h_2)$. Since J_i is a diffeomorphism, the map I must be injective and hence a global isometry onto its image. ■

Having established that limits arising from different radii are related via an isometry, we see that one may be viewed as a subset of the other. In proving Theorem 3.6.1 we will want to relate a countably infinite number of related limit manifolds, each of which can be isometrically embedded into the next. The following result records how to construct a single smooth manifold, into which each limit can be embedded in such manner that the image of the i^{th} limit is contained within the image of the $(i+1)^{th}$.

Theorem 3.3.2 (Smooth Manifold Construction; Theorem C.1 in [MT18])**.** *Assume that for each $i \in \mathbb{N}$ we have a smooth n-manifold $\mathcal{M}_i$ and a point $x_i \in \mathcal{M}_i$, and that each $\mathcal{M}_i$ is contained in the next in the sense that for each $i \in \mathbb{N}$ there exists a smooth map $\psi_i : \mathcal{M}_i \to \mathcal{M}_{i+1}$, mapping x_i*

to x_{i+1} and diffeomorphic onto its image. Then there exists a smooth n-manifold M, containing a point x_0, and there exist smooth maps $\theta_i : \mathcal{M}_i \to M$, all mapping x_i to x_0, diffeomorphic onto their image, and satisfying that $\theta_i(\mathcal{M}_i) \subset \theta_{i+1}(\mathcal{M}_{i+1})$, and further that

$$M = \bigcup_{i=1}^{\infty} \theta_i(\mathcal{M}_i). \tag{3.3.1}$$

Moreover, we have that

$$\psi_i = \theta_{i+1}^{-1} \circ \theta_i : \mathcal{M}_i \to \mathcal{M}_{i+1}. \tag{3.3.2}$$

Proof. Define $M := \bigsqcup_{i=1}^{\infty} \mathcal{M}_i / \sim$, equipped with the quotient topology, where $\sim$ is the equivalence relation generated by identifying points x and y if $y = \psi_i(x)$ for some $i \in \mathbb{N}$. Let $x_0 \in M$ be the equivalence class generated by the points $x_i \in \mathcal{M}_i$. For each $i \in \mathbb{N}$ define $\theta_i : \mathcal{M}_i \to M$ to be the map sending a point x to the equivalence class $[x]$. Thus $\theta_i(x_i) = x_0$, and θ_i is a homeomorphism onto its image, while $M = \bigcup_{i=1}^{\infty} \theta_i(\mathcal{M}_i)$ which is (3.3.1). Moreover, for each $x \in \mathcal{M}_i$ we have $\theta_i(x) = [x] = [\psi_i(x)] = \theta_{i+1}(\psi_i(x))$, which gives (3.3.2). Since ψ_i is a diffeomorphism onto its image, (3.3.2) allows us to combine the smooth atlases for each $\mathcal{M}_i$ into a smooth atlas for M by composing with the maps θ_i^{-1}. Hence we simultaneously establish both that M is a smooth n-manifold, and that each θ_i is a diffeomorphism onto its image as claimed. ∎

Throughout we will need to pay particular attention to how the distance functions behave under the local convergence. The following result records several useful properties that we will later require.

Lemma 3.3.3 (Distance function convergence under local convergence; Lemma 6.1 in [ST17])**.** *Suppose $(\mathcal{M}_i^n, g_i)$ is a sequence of smooth $n-$dimensional Riemannian manifolds, possibly incomplete, and $x_i \in \mathcal{M}_i$ for each i. Suppose there exist a, possibly incomplete, smooth Riemannian n-manifold $(\mathcal{N}, h)$ and a point $x_0 \in \mathcal{N}$ with $\mathbb{B}_h(x_0, 2r) \subset\subset \mathcal{N}$ for some $r > 0$, and a sequence of smooth maps $\varphi_i : \mathcal{N} \to \mathcal{M}_i$, diffeomorphic onto their images, with $\varphi_i(x_0) = x_i$ for each i, such that $\varphi_i^* g_i \to h$ smoothly on $\overline{\mathbb{B}_h(x_0, 2r)}$. Then*

1. *If $0 < a \leq 2r$, and $a < b$, then $\varphi_i\left(\mathbb{B}_h(x_0, a)\right) \subset \mathbb{B}_{g_i}(x_i, b)$ for sufficiently large i.*
2. *If $0 < a < b \leq 2r$, then $\mathbb{B}_{g_i}(x_i, a) \subset\subset \varphi_i\left(\mathbb{B}_h(x_0, b)\right)$ for sufficiently large i.*
3. *For every $s \in (0, r)$, we have*

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d_h(x, y)$$

as $i \to \infty$, uniformly for x and y in $\mathbb{B}_h(x_0, s)$.

3.4. Fixed Radius Compactness Implies Theorem 3.2.1

In this section we prove Theorem 3.2.1 under the assumption that the following result is true.

Lemma 3.4.1 (Compactness at fixed $r \in (0, R)$)**.** *Suppose $(\mathcal{M}_i^n, g_i)$ is a sequence of smooth n-dimensional Riemannian manifolds, not necessarily complete, and that $x_i \in \mathcal{M}_i$ for each i. Suppose that, for some $R > 0$, we have $\mathbb{B}_{g_i}(x_i, R) \subset\subset \mathcal{M}_i$ for each i, that $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, R) \geq v > 0$ and that (for each l) we have $|\nabla^l \mathrm{Rm}|_{g_i} \leq C_l$ throughout $\mathbb{B}_{g_i}(x_i, R)$, for constants C_l and v that are independent of i (with C_l allowed to depend on l).*

Then for a fixed $r \in (0, R)$, after passing to a subsequence in i, there exists a smooth n-dimensional Riemannian manifold $(\mathcal{N}_r, g_\infty^r)$, a point $x_r \in \mathcal{N}_r$ with $\mathbb{B}_{g_\infty^r}(x_r, r) \subset\subset \mathcal{N}_r$ and a sequence of smooth maps $F_i^r : \mathcal{N}_r \to \mathcal{M}_i$, diffeomorphic onto their images, mapping x_r to x_i and satisfying that $(F_i^r)^* g_i \to g_\infty^r$ smoothly uniformly on $\overline{\mathbb{B}_{g_\infty^r}(x_r, r)}$.

We now illustrate how this lemma allows us to establish Theorem 3.2.1.

Proof of Theorem 3.2.1 (assuming Lemma 3.4.1). For each $j \in \mathbb{N}$ define $r_j := \frac{j}{j+1} R \in (0, R)$. We may appeal to Lemma 3.4.1 for $r := r_j$ to obtain, after passing to a subsequence in i, a smooth n-dimensional Riemannian manfiold $(\mathcal{N}_{r_j}, g_{r_j})$, with a point $x_{r_j} \in \mathcal{N}_{r_j}$ such that $\mathcal{B}_j := \mathbb{B}_{g_{r_j}}(x_{r_j}, r_j) \subset\subset \mathcal{N}_{r_j}$, and a sequence of smooth maps $F_i^j : \mathcal{N}_{r_j} \to \mathcal{M}_i$, diffeomorphic onto their image, mapping x_{r_j} to x_i and satisfying $(F_i^j)^* g_i \to g_{r_j}$ smoothly uniformly on $\overline{\mathcal{B}_j}$.

By taking an appropriate diagonal subsequence in i, we can be sure that these limits exist for *every* $j \in \mathbb{N}$. We now wish to relate the limit metrics g_{r_j} that we have constructed, for different j. Let us fix $j \in \mathbb{N}$. Then g_{r_j} is the smooth limit of the metrics g_i (modulo the diffeomorphisms F_i^j) defined on $\overline{\mathcal{B}_j}$. On the other hand, $g_{r_{j+1}}$ is the smooth limit of the metrics g_i (modulo the diffeomorphisms F_i^{j+1}) defined on $\overline{\mathcal{B}_{j+1}}$. Since $r_{j+1} > r_j$ we intuitively expect $\mathcal{B}_{j+1}$ to be "bigger" than $\mathcal{B}_j$. This intuition is made precise in the following claim.
Claim: For sufficiently large i we have

$$F_i^j(\mathcal{B}_j) \subset\subset F_i^{j+1}(\mathcal{B}_{j+1}) \subset\subset \mathcal{M}_i. \tag{3.4.1}$$

Indeed, we have the stronger inclusion that for sufficiently large i, depending on j,

$$F_i^j(\mathcal{B}_j) \subset\subset F_i^{j+1}\left(\mathbb{B}_{g_{r_{j+1}}}\left(x_{r_{j+1}}, r_j + \frac{\eta}{2}\right)\right), \tag{3.4.2}$$

where $\eta := r_{j+1} - r_j > 0$, which immediately yields the first inclusion in (3.4.1)
Proof: We have both that $(F_i^j)^* g_i \to g_{r_j}$ smoothly uniformly on $\overline{\mathcal{B}_j}$ and that $(F_i^{j+1})^* g_i \to g_{r_{j+1}}$ smoothly uniformly on $\overline{\mathcal{B}_{j+1}}$ as $i \to \infty$.

Let $\eta := r_{j+1} - r_j > 0$ and $\rho := R - r_{j+1} > 0$. Appealing to both Parts 1 and 2 of Lemma 3.3.3, taking $(\mathcal{N}, \hat{g})$, $x_0 \in \mathcal{N}$ and $2r$ there as $(\mathcal{N}_{r_{j+1}}, g_{r_{j+1}})$, $x_{r_{j+1}} \in \mathcal{N}_{r_{j+1}}$ and r_{j+1} here respectively, allows us to deduce that for sufficiently large i we have

$$\mathbb{B}_{g_i}\left(x_i, r_j + \frac{\eta}{4}\right) \subset\subset F_i^{j+1}\left(\mathbb{B}_{g_{r_{j+1}}}\left(x_{r_{j+1}}, r_j + \frac{\eta}{2}\right)\right) \subset F_i^{j+1}(\mathcal{B}_{j+1}) \subset \mathbb{B}_{g_i}\left(x_i, r_{j+1} + \frac{\rho}{2}\right). \tag{3.4.3}$$

Since $r_{j+1} + \frac{\rho}{2} < R$ we may conclude from (3.4.3) that $\mathbb{B}_{g_i}\left(x_i, r_{j+1} + \frac{\rho}{2}\right) \subset\subset \mathbb{B}_{g_i}(x_i, R) \subset\subset M_i$ and hence we obtain the second inclusion required in (3.4.1)

We may appeal to Part 1 of Lemma 3.3.3, this time taking $(\mathcal{N}, \hat{g})$, $x_0 \in \mathcal{N}$ and $2r$ there as $(\mathcal{N}_{r_j}, g_{r_j})$, $x_{r_j} \in \mathcal{N}_{r_j}$ and r_j here respectively, to conclude that for sufficiently large i we have

$$F_i^j(\mathcal{B}_j) \subset \mathbb{B}_{g_i}\left(x_i, r_j + \frac{\eta}{4}\right). \tag{3.4.4}$$

Combining (3.4.3) and (3.4.4) yields (3.4.2), which itself establishes the first inclusion required in (3.4.1). ††

The claim allows us to apply Lemma 3.3.1 with $(\mathcal{N}_1, h_1) = (\mathcal{N}_{r_j}, g_{r_j})$, $O_1 := \mathcal{B}_j$ and $(\mathcal{N}_2, h_2) = (\mathcal{N}_{r_{j+1}}, g_{r_{j+1}})$, $O_2 := \mathcal{B}_{j+1}$ to obtain a smooth map $I_{r_j} : \mathcal{B}_j \to \mathcal{B}_{j+1}$, mapping x_{r_j} to $x_{r_{j+1}}$, and giving an isometry onto its images with respect to the metrics g_{r_j} and $g_{r_{j+1}}$.

Indeed, after passing to another subsequence in i, we could see I_{r_j} as a smooth limit, as $i \to \infty$, of maps $\left(F_i^{j+1}\right)^{-1} \circ F_i^j$ which are well-defined thanks to the claim. Seeing I_{r_j} as such a limit and appealing to (3.4.2) allows us to conclude that

$$I_{r_j}(\mathcal{B}_j) \subset\subset \mathbb{B}_{g_{r_{j+1}}}\left(x_{r_{j+1}}, r_j + \frac{\eta}{2}\right) \tag{3.4.5}$$

where as before $\eta := r_{j+1} - r_j > 0$.

The isometries I_{r_j} allow us to appeal to Lemma 3.3.2 to obtain a smooth n-dimensional manifold $\mathcal{N}$, a point $x_0 \in \mathcal{N}$, smooth maps $\theta_j : \mathcal{B}_j \to \mathcal{N}$, mapping x_{r_j} to x_0, diffeomorphic onto their image, satisfying that $\theta_j(\mathcal{B}_j) \subset \theta_{j+1}(\mathcal{B}_{j+1})$, with the compatibility $I_{r_j} = \theta_{j+1}^{-1} \circ \theta_j$ and the decomposition $\mathcal{N} = \bigcup_{j \in \mathbb{N}} \theta_j(\mathcal{B}_j)$.

We can consider the pull-back metric $(\theta_j^{-1})^* g_{r_j}$ on $\theta_j(\mathcal{B}_j) \subset \mathcal{N}$ for each j, and because I_{r_j} is an isometry, these pull-backs agree where they overlap. The union of the pull-backs we call g_∞. We now strengthen the inclusion $\theta_j(\mathcal{B}_j) \subset \theta_{j+1}(\mathcal{B}_{j+1})$ to assert that the images of $\mathcal{B}_j$ are contained within the interior of $\mathcal{N}$.

Since $I_{r_j} = \theta_{j+1}^{-1} \circ \theta_j$, (3.4.5) implies that $\theta_{j+1}^{-1}\left(\theta_j(\mathcal{B}_j)\right) \subset\subset \mathbb{B}_{g_{r_{j+1}}}\left(x_{r_{j+1}}, r_j + \frac{\eta}{2}\right) \subset\subset$

$\mathcal{B}_{j+1}$. Therefore we can strengthen $\theta_j(\mathcal{B}_j) \subset \theta_{j+1}(\mathcal{B}_{j+1})$ to

$$\theta_j(\mathcal{B}_j) \subset\subset \mathbb{B}_{g_\infty}\left(x_0, r_j + \frac{\eta}{2}\right) \subset\subset \theta_{j+1}(\mathcal{B}_{j+1}), \tag{3.4.6}$$

establishing that $\theta_j(\mathcal{B}_j)$ is contained within the interior of $\mathcal{N}$. Therefore we may conclude that $\theta_j(\mathcal{B}_j) \subset\subset \mathcal{N}$ for every $j \in \mathbb{N}$. Since $\theta_j(\mathcal{B}_j) = \mathbb{B}_{g_\infty}(x_0, r_j)$, and the sequence $r_j \uparrow R$ as $j \to \infty$, we may conclude that for every $s \in (0, R)$ we have $\mathbb{B}_{g_\infty}(x_0, s) \subset\subset \mathcal{N}$ as required.

For each $j \in \mathbb{N}$ we have a sequence

$$f_i^j : \theta_j(\mathcal{B}_j) \to \mathcal{M}_i \tag{3.4.7}$$

of smooth maps, for $i \geq j$, defined by $f_i^j := F_i^j \circ \theta_j^{-1}$, that map x_0 to x_i and are diffeomorphic onto their images. Moreover, from our choice of diagonal subsequence, we have that

$$(f_i^j)^* g_i \to g_\infty \tag{3.4.8}$$

smoothly uniformly on $\theta_j(\mathcal{B}_j) = \mathbb{B}_{g_\infty}(x_0, r_j)$.

We now turn our attention to defining the smooth maps φ_i. For each $j \in \mathbb{N}$ we can appeal to the smooth convergence established in (3.4.8) to choose $I(j)$ such that for all $i \geq I(j)$ and any $p \in \{0, \dots, j\}$ we have

$$\left| \nabla^p_{g_\infty} \left((f_i^j)^* g_i - g_\infty \right) \right|_{g_\infty} \leq \frac{1}{j} \tag{3.4.9}$$

throughout $\theta_j(\mathcal{B}_j) = \mathbb{B}_{g_\infty}(x_0, r_j)$. We may assume that $I(j)$ is strictly increasing in j, otherwise we can fix $I(1)$ and then inductively replace $I(j)$ for $j = 2, 3, \dots$ by the maximum of $I(j)$ and $I(j-1) + 1$. Pass to a further subsequence in i by selecting the entries $I(1), I(2), I(3), \dots$, so that the estimates in (3.4.9) now hold for all $i \geq j$.

The sequence of smooth maps $\varphi_i := f_i^i$ for $i \in \mathbb{N}$ are our candidates to give the required diffeomorphisms. As required $\varphi_i : \mathbb{B}_{g_\infty}\left(x_0, \frac{i}{i+1}R\right) \to \mathcal{M}_i$ is a smooth map, mapping x_0 to x_i and diffeomorphic onto its image. To conclude, we need only establish that $\varphi_i^* g_i \to g_\infty$ smoothly locally on $\mathcal{N}$. To do so, it suffices to establish that $\varphi_i^* g_i \to g_\infty$ smoothly on $\mathbb{B}_{g_\infty}(x_0, r_j)$ for every $j \in \mathbb{N}$, which is immediate from (3.4.9). ∎

3.5. Proof of Lemma 3.4.1

We need only establish Lemma 3.4.1 to complete the proof of Theorem 3.2.1. For this we will follow the method of Kasue in [Kas89] which uses *harmonic coordinates*. The properties of harmonic coordinates we need are summarised below. This particular formulation is based on

Fact 1.1 in [Kas89], though the results originally appear earlier in [Jos83] and [GW88]. We have explicitly recorded an additional property ((3.5.3)) compared with Fact 1.1 in [Kas89] that is useful for our purposes, and is already implicit in the construction of harmonic coordinates in [Jos83].

Lemma 3.5.1 (Harmonic coordinates; Variant of Fact 1.1 in [Kas89])**.** *Let $(\mathcal{M}, G)$ be a smooth n-dimensional Riemannian manifold with a point $p \in \mathcal{M}$ for which $\mathbb{B}_G(p, r) \subset\subset \mathcal{M}$ and $\mathrm{inj}_G(p) \geq I > 0$. Further, assume that for each $l \in \mathbb{N}$ we have $|\nabla^l \mathrm{Rm}|_G \leq C_l$ throughout $\mathbb{B}_G(p, r)$.*

Then there exists a constant $\delta_0 = \delta_0(n, C_0, I, r) > 0$ and a harmonic map $H = (h_1, \ldots, h_n) : \mathbb{B}_G(p, \delta_0) \to \mathbb{R}^n$, mapping p to 0, defining a coordinate system around p which has the following properties:

$$(1 + \eta_0(n, C_0, I, r))^{-1} d_G(p, x) \leq |H(x)| \leq (1 + \eta_0(n, C_0, I, r)) d_G(p, x) \tag{3.5.1}$$

$$(1 + \eta_0(n, C_0, I, r))^{-1} |\xi|^2 \leq g_{ij}(x) \xi^i \xi^j \leq (1 + \eta_0(n, C_0, I, r)) |\xi|^2 \tag{3.5.2}$$

$$\text{For every } x \in \mathbb{B}_G(p, \delta_0) \qquad |\det [DH(x)]| \geq \eta_1(n, C_0, I, r) > 0 \tag{3.5.3}$$

$$||g_{ij}||_{C^{1+k,\beta}(\mathbb{B}_G(p,\delta_0))} \leq \eta_2(n, C_0, \ldots, C_k, I, \beta, r) \quad (0 < \beta < 1, k \in \mathbb{N}_0) \tag{3.5.4}$$

where we set $g_{ij} := G\left(\frac{\partial}{\partial h_i}, \frac{\partial}{\partial h_j}\right)$ and the norms are taken in the h_i coordinates. Further, for any harmonic function f on $\mathbb{B}_G(p, \delta_0)$ and every $k \in \mathbb{N}_0$ we have

$$||f||_{C^{2+k,\beta}\left(\mathbb{B}_G\left(p, \frac{\delta_0}{2}\right)\right)} \leq \eta_3(n, C_0, \ldots, C_k, I, \beta, r) \sup_{\mathbb{B}_G(p,\delta_0)} \{|f|\} \tag{3.5.5}$$

where the $(2 + k, \beta)$ norm is taken in the h_i coordinates.

A potential approach to proving Lemma 3.4.1 is to first use a compactness result of Gromov (see Theorem 7.4.15 in [BBI01]) to pass to a subsequence in i and obtain convergence to a limit metric space (X, d, x_0) in the pointed Gromov-Hausdorff sense. By covering each element of the sequence with harmonic coordinate balls arising from Lemma 3.5.1 we can first work locally within the harmonic coordinate patches. Roughly, each harmonic coordinate patch can be identified with a region of X via a locally defined diffeomorphism. The metrics g_i can be pulled back to X via this local map, and the regularity provided by Lemma 3.5.1 allows us to pass to a subsequence in i and obtain smooth convergence of the metrics g_i to a smooth limit within this patch. Repeating for each harmonic coordinate patch, passing to successive subsequences in i, we may obtain such smooth convergence within each patch simultaneously. These local statements may be patched together to obtain a smooth Riemannian metric h, defined globally throughout X, and arising as the smooth limit of the metrics g_i, see [Pet97] for full details.

Constructing the required diffeomorphisms is where the majority of the work is required.

In [Pet97] they are constructed via a weighted averaging of the local diffeomorphisms obtained in each harmonic coordinate patch. The construction of the diffeomorphisms may be streamlined by following the ideas of Kasue in [Kas89]. In Theorem A in [Kas89], Kasue uses harmonic coordinates to construct smooth embeddings of the elements $\mathcal{M}_i$ into some Euclidean space $\mathbb{R}^N$, for a fixed N independent of i. Being able to work with C^∞ embedded submanifolds within $\mathbb{R}^N$ then allows the diffeomorphisms to be constructed via normal projections.

In order to exploit this simplification we follow the argument of Theorem A in [Kas89], making the appropriate technical alterations to deal with the elements of our sequence $\mathcal{M}_i$ no longer being assumed to be compact. In places we closely follow the arguments of Miles Simon in Appendix B of [Sim15] where a four dimensional compactness result (with weaker curvature hypotheses) is obtained.

Proof of Lemma 3.4.1. Recall that we have a sequence $(\mathcal{M}_i, g_i)$ of smooth (possibly incomplete) n-dimensional Riemannian manifolds, with points $x_i \in \mathcal{M}_i$ for which $\mathbb{B}_{g_i}(x_i, R) \subset\subset \mathcal{M}_i$, as well as having both the volume lower bound that $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, R) \geq v > 0$ and the curvature estimates that for every $l \in \mathbb{N}_0$ we have $|\nabla^l \mathrm{Rm}|_{g_i} \leq C_l$ throughout $\mathbb{B}_{g_i}(x_i, R)$.

Fix $r \in (0, R)$, define $\rho := \frac{R-r}{2} > 0$ and choose $s := r + \rho \in (r, R)$ so that we also have that $R - s = \rho > 0$. For $i \in \mathbb{N}$ we simplify notation by defining $B_i := \mathbb{B}_{g_i}(x_i, s)$. The lower bounds on the Ricci tensor throughout $\mathbb{B}_{g_i}(x_i, R)$ and on the volume of $\mathbb{B}_{g_i}(x_i, R)$ allow us to appeal to the Bishop-Gromov comparison theorem via Lemma 2.2.3 to reduce $v > 0$, depending only on n, C_0, v, R and ρ, so that for all $x \in B_i$ we have $\mathrm{Vol}\mathbb{B}_{g_i}(x, \rho) \geq v > 0$. Combined with the curvature bounds these lower volume bounds allow us to conclude, via the Cheeger-Gromov-Taylor injectivity radius estimates of Theorem 2.7.3, that for some $I = I(n, C_0, v, R, \rho) > 0$ we have $\mathrm{inj}_{g_i}(x) \geq I$ for all $x \in B_i$.

Given any $z \in B_i$ we know that $\mathbb{B}_{g_i}\left(z, \frac{\rho}{2}\right) \subset\subset \mathbb{B}_{g_i}(x, R) \subset\subset \mathcal{M}_i$, which immediately gives $|\nabla^l \mathrm{Rm}|_{g_i} \leq C_l$ throughout $\mathbb{B}_{g_i}\left(z, \frac{\rho}{2}\right)$, and that $\mathrm{inj}_{g_i}(z) \geq I$. Therefore we may appeal to Lemma 3.5.1 to obtain harmonic coordinates on the ball $\mathbb{B}_{g_i}(z, \delta_0)$ for a constant $\delta_0 > 0$ depending only on n, C_0, v, R and ρ. For later use we observe that we may additionally assume that δ_0 is taken sufficiently small to guarantee that $\delta_0 < \min\left\{1, \frac{\rho}{2}\right\}$.

Having obtained the constant $\delta_0 > 0$, we let η_0, η_1 and η_2 be the respective constants arising from Lemma 3.5.1. We now wish to obtain a cover of B_i with harmonic coordinate patches. Throughout the proof it will be convenient to restrict to smaller radii balls, hence we will choose a collection of harmonic coordinate patches so that the balls of much smaller radii than δ_0 still provide a cover of B_i. For this purpose, we introduce the following monotonically increasing family of constants $\delta_k \in (0, \delta_0)$ for $k = 1, ..., 9$. Being precise, we define $\delta_1 := 10^{-1}(1+\eta_0)^{-9}\delta_0$ and $\delta_k := (1 + \eta_0)^k \delta_1$ for $k \in \{2, 3, 4, 5, 6, 7, 8, 9\}$. The reasoning for defining this number of constants will become apparent during the proof.

With these constants defined, we aim to choose a finite subset $\Gamma := \{p_i^1, \ldots, p_i^\mu\} \subset B_i$ such that given any $z \in B_i$ there exists some p_i^m for which $d_{g_i}(p_i^m, z) \leq \delta_1$, and that if $m \neq k$ then $d_{g_i}(p_i^m, p_i^k) > \frac{\delta_1}{2}$. That this is possible with a uniformly (in i) bounded number of points p_i^m is a consequence of volume comparison. Indeed, Bishop-Gromov volume comparison theorem 2.2.1 (in particular, the first consequence stated in (2.2.2)) yields that $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, R) \leq c_1(n, C_0, R)$. Now assume that the set $\{p_i^1, \ldots, p_i^\mu\} \subset B_i$ for some $\mu \in \mathbb{N}$ gives such a desired set. For each $m \in \{1, \ldots, \mu\}$ we may first use the Bishop-Gromov theorem 2.2.1, with $H := -C_0$ and recalling that $\delta_1 < \delta_0 < \rho$, to conclude that

$$\mathrm{Vol}\mathbb{B}_{g_i}\left(p_i^m, \frac{\delta_1}{2}\right) \geq \frac{\mathrm{Vol}\mathbb{B}_{g_H}\left(p_H, \frac{\delta_1}{2}\right)}{\mathrm{Vol}\mathbb{B}_{g_H}(p_H, \rho)} \mathrm{Vol}\mathbb{B}_{g_i}(p_i^m, \rho) \geq c_2(n, C_0, v, R, \rho). \tag{3.5.6}$$

Therefore we can compute that

$$c_2(n, C_0, v, R, \rho)\mu \overset{(3.5.6)}{\leq} \sum_{m=1}^{\mu} \mathrm{Vol}\mathbb{B}_{g_i}\left(p_i^m, \frac{\delta_1}{2}\right) \leq \mathrm{Vol}\mathbb{B}_{g_i}(x_i, R) \leq c_1.$$

Hence μ, the number of points required, is bounded above by a constant $\mu_{\delta_1} = \mu_{\delta_1}(n, C_0, v, R, \rho)$. By passing to a subsequence in i, we may assume that the same number of points μ is required for each $i \in \mathbb{N}$, where μ is a constant $\mu \leq \mu_{\delta_1}$ that is independent of i.

For each $m \in \{1, \ldots, \mu\}$ we let H_i^m denote the harmonic diffeomorphism inducing a coordinate system on $\mathbb{B}_{g_i}(p_i^m, \delta_0)$. From (3.5.1) and (3.5.2) of Lemma 3.5.1 we have

$$\mathbb{B}^n(0, (1+\eta_0)^{-1}a) \subset H_i^m\left(\mathbb{B}_{g_i}(p_i^m, a)\right) \subset \mathbb{B}^n(0, (1+\eta_0)a) \tag{3.5.7}$$

for any $a \in (0, \delta_0)$ where $\mathbb{B}^n(0, u) := \{x \in \mathbb{R}^n : |x| < u\}$.

We now use the harmonic diffeomorphisms H_i^m to construct a C^∞ smooth map $\mathscr{H}_i : \Sigma_i \to \mathbb{R}^N$ where

$$\Sigma_i := \bigcup_{m=1}^{\mu} \mathbb{B}_{g_i}(p_i^m, \delta_0) \tag{3.5.8}$$

and $N \in \mathbb{N}$ depends only on n, C_0, v, R and ρ. We also note that $\Sigma_i \subset\subset \mathbb{B}_{g_i}(x_i, R)$, which follows since $\delta_0 < \frac{\rho}{2}$. The map $\mathscr{H}_i$ will later be used to smoothly embed a subset of Σ_i in $\mathbb{R}^N$.

In order to extend each H_i^m to the whole of Σ_i we require a smooth cut-off function. This cut-off function will also play a role in ensuring the map $\mathscr{H}_i$ is injective on some subset of Σ_i. Roughly, it will be used to smoothly shift the images of each of the balls $\mathbb{B}_{g_i}(p_i^m, \delta_0)$ to avoid self-intersections, analogously to the Whitney embedding theorem. This will be made precise below.

First let $\xi : [0, \infty) \to [0, \infty)$ be a smooth non-increasing cut-off function such that

$$\begin{cases} \xi \equiv 1 & \text{on } [0, \delta_7] \\ \xi \equiv 0 & \text{on } [\delta_8, \infty). \end{cases} \tag{3.5.9}$$

Then for each $m \in \{1, \ldots, \mu\}$ the function $\xi_i^m : \mathbb{B}_{g_i}(p_i^m, \delta_0) \to \mathbb{R}$ given by $\xi_i^m(z) := \xi\left(|H_i^m(z)|\right)$ is smooth, and in particular vanishes when $d_{g_i}(p_i^m, z) \geq (1 + \eta_0)\delta_8$. Hence each ξ_i^m may be smoothly extended (by zero) to the whole of Σ_i. Now define $\mathscr{H}_i : \Sigma_i \to \mathbb{R}^N$, with $N := (n+1)\mu \in \mathbb{N}$, by

$$\mathscr{H}_i(z) := \left(\xi_i^1(z)H_i^1(z), \ldots, \xi_i^\mu(z)H_i^\mu(z), \xi_i^1(z), \ldots, \xi_i^\mu(z)\right), \tag{3.5.10}$$

where for each $m \in \{1, \ldots, \mu\}$ we understand the product $\xi_i^m H_i^m$ to be extended smoothly by 0 to be defined on the entirety of Σ_i. Since each ξ_i^m vanishes outside $\overline{\mathbb{B}_{g_i}(p_i^m, (1+\eta_0)\delta_8)}$ and $(1+\eta_0)\delta_8 < \frac{\delta_0}{2}$, the regularity provided by Lemma 3.5.1 (in particular, (3.5.5)) tells us that $\mathscr{H}_i$ is smooth, and we may choose a sequence $K_l > 0$, defined for $l \in \mathbb{N}_0$, depending only on n, $C_0, C_1, \ldots, C_{l-2}$, v, R, and ρ, such that for every $l \in \mathbb{N}_0$ we have that

$$||\mathscr{H}_i||_{C^l(\Sigma_i; g_i)} \leq K_l. \tag{3.5.11}$$

To clarify, if $l = 0, 1, 2$ then there is only dependence on C_0 (along with the other constants) whilst for $l \geq 3$ there is dependence on all constants $C_0, C_1, \ldots, C_{l-2}$ (along with the other constants). The $l = 0$ case of (3.5.11) tells us that $\mathscr{H}_i(B_i) \subset\subset \mathbb{B}^N(0, R_0)$, where $R_0 := K_0 + 1$ depends only on n, C_0, v, R and ρ.

With a view to later writing the embedded images in $\mathbb{R}^N$ of restrictions of the maps $\mathscr{H}_i$ as unions of graphs, we make the following observations regarding the regularity of H_i^m for $m \in \{1, \ldots, \mu\}$. Momentarily fix $m \in \{1, \ldots, \mu\}$. Since H_i^m is a diffeomorphism we know its inverse $(H_i^m)^{-1}$ is smooth. Moreover, (3.5.7) tells us that

$$H_i^m\left(\mathbb{B}_{g_i}\left(p_i^m, \frac{\delta_0}{4}\right)\right) \supset \mathbb{B}^n\left(0, \frac{\delta_0}{4(1+\eta_0)}\right) \supset \overline{\mathbb{B}^n(0, \delta_8)}. \tag{3.5.12}$$

Therefore $(H_i^m)^{-1}$ is defined throughout $\overline{\mathbb{B}^n(0, \delta_8)}$.

We would like to consider the transition functions given by $H_i^k \circ (H_i^m)^{-1}$ for $m, k \in \{1, \ldots, \mu\}$. Such functions will only be defined throughout $H_i^m\left(\mathbb{B}_{g_i}(p_i^m, \delta_0) \cap \mathbb{B}_{g_i}(p_i^k, \delta_0)\right)$, and there is no reason that H_i^k must be defined throughout $\mathbb{B}_{g_i}(p_i^m, \delta_0)$. However, recalling our extension above, both ξ_i^k and the product $\xi_i^k H_i^k$ are defined throughout the whole of Σ_i, and so in particular throughout $\mathbb{B}_{g_i}(p_i^m, \delta_0)$. Therefore, for every $i \in \mathbb{N}$ and all pairs $m, k \in \{1, \ldots, \mu\}$, we can consider the functions $F_i^{k,m} := \xi_i^k H_i^k \circ (H_i^m)^{-1}$ and $f_i^{k,m} := \xi\left(|H_i^k \circ (H_i^m)^{-1}|\right)$, where $f_i^{k,m}$ is smoothly extended by 0, so that both exist throughout $H_i^m\left(\mathbb{B}_{g_i}(p_i^m,\delta_0)\right)$, and so in particular, by (3.5.12), throughout $\overline{\mathbb{B}^n(0,\delta_8)}$.

Consider a fixed $m \in \{1,\ldots,\mu\}$. From (3.5.3) of Lemma 3.5.1 we have that $\det\left[DH_i^m\right]$ is uniformly (in i) bounded away from 0. The chain rule and the formula for matrix inversion allows us to conclude that for $l,q \in \{1,\ldots,n\}$

$$\frac{\partial\left(\left(\left(H_i^m\right)^{-1}\right)_q\right)}{\partial x_l}\left(H_i^m(x)\right) = \frac{(\text{Polynomial in components of } DH_i^m(x))}{\det\left[DH_i^m(x)\right]} \tag{3.5.13}$$

where $\left(\left(H_i^m\right)^{-1}\right)_q$ denotes the q^{th} component (in the h_i coordinates) of $\left(H_i^m\right)^{-1}$. The formulae in (3.5.13), combined with the C^1 bounds for H_i^m arising from (3.5.5) of Lemma 3.5.1, allow us to deduce uniform (in i) estimates on $D\left(H_i^m\right)^{-1}$ throughout, in particular, $\overline{\mathbb{B}^n(0,\delta_8)}$.

By differentiating the formulae in (3.5.13) and recalling the determinant estimate (3.5.3) in Lemma 3.5.1, we see that the regularity of $\mathscr{H}_i$ given in (3.5.11) and the inclusions of (3.5.12) allow us to obtain uniform (in i) C^l estimates, for every $l \in \mathbb{N}$, on $\left(H_i^m\right)^{-1}$ over $\overline{\mathbb{B}^n(0,\delta_8)}$. By repeating for each $m \in \{1,\ldots,\mu\}$ we may deduce such estimates for $\left(H_i^m\right)^{-1}$ for all $m \in \{1,\ldots,\mu\}$.

The aforementioned regularity obtained for each $\left(H_i^m\right)^{-1}$ over $\overline{\mathbb{B}^n(0,\delta_8)}$, combined with the estimates of (3.5.11), allow us to conclude that for any $i \in \mathbb{N}$ and any $k,m \in \{1,\ldots,\mu\}$ we have

$$||F_i^{k,m}||_{C^l(\mathbb{B}^n(0,\delta_8);\mathbb{R}^n)}, ||f_i^{k,m}||_{C^l(\mathbb{B}^n(0,\delta_8);\mathbb{R})} \leq \mathcal{A}_l(l,n,C_0,\ldots,C_{l-2},v,R,\rho) \tag{3.5.14}$$

for every $l \in \mathbb{N}$. We can appeal to the Ascoli-Arzelà theorem to obtain useful convergence properties for the functions $F_i^{k,m}$ and $f_i^{k,m}$, for $k,m \in \{1,\ldots,\mu\}$, on, in particular, $\mathbb{B}^n(0,\delta_7)$ as $i \to \infty$ Indeed, the bounds obtained in (3.5.14) are independent of i, which allows us to appeal to the Ascoli-Arzelà theorem and conclude that, for every $k,m \in \{1,\ldots,\mu\}$, there exists smooth functions $F^{k,m} \in C^\infty(\mathbb{B}^n(0,\delta_7);\mathbb{R}^n)$ and $f^{k,m} \in C^\infty(\mathbb{B}^n(0,\delta_7);\mathbb{R})$ such that, after passing to a subsequence in i, we have both $F_i^{k,m} \to F^{k,m}$ and $f_i^{k,m} \to f^{k,m}$ smoothly uniformly on $\mathbb{B}^n(0,\delta_7)$ as $i \to \infty$. By repeatedly passing to a subsequence in i, we may assume this convergence is valid for all $m,k \in \{1,\ldots,\mu\}$ simultaneously, and we may observe that the estimates of (3.5.14), over the ball $\mathbb{B}^n(0,\delta_7)$, pass to the limit.

At this point we could try to use the functions $F_i^{k,m}$ and $f_i^{k,m}$ to write $\mathscr{H}_i(\Sigma_i)$ as a union of graphs. However, in order to ensure the image has no self intersections, we first restrict to a subset of Σ_i before considering the image under $\mathscr{H}_i$. That is, consider the subset $\tilde{\Sigma}_i \subset\subset \Sigma_i \subset\subset \mathbb{B}_{g_i}(x_i,R)$ defined by

$$\tilde{\Sigma}_i := \bigcup_{m=1}^{\mu}\left(H_i^m\right)^{-1}\left(\mathbb{B}^n(0,\delta_7)\right). \tag{3.5.15}$$

Suppose $\mathscr{H}_i(z) = \mathscr{H}_i(w)$ for some $z, w \in \tilde{\Sigma}_i$. From (3.5.15) we know that $z = (H_i^m)^{-1}(a)$ for some $m \in \{1, \ldots, \mu\}$ and $a \in \mathbb{B}^n(0, \delta_7)$. Obviously we have that $|H_i^m(z)| = |a| < \delta_7$ and so $\xi_i^m(z) = 1$. Therefore we must also have that $\xi_i^m(w) = 1$, and so w must also belong to the domain of H_i^m. Moreover, since both $\xi_i^m(z) = 1 = \xi_i^m(w)$ we see from (3.5.10) that $H_i^m(z) = H_i^m(w)$. But H_i^m is a diffeomorphism, hence we must have that $z = w$ which establishes the claimed injectivity.

Further, $\mathscr{H}_i$ is an immersion on $\tilde{\Sigma}_i$, as we will show below. First note that given any $z \in \tilde{\Sigma}_i$ there is some $m \in \{1, \ldots, \mu\}$ such that $H_i^m(z) \in \mathbb{B}^n(0, \delta_7)$. In particular, $|H_i^m(z)| < \delta_7$ and so, by continuity, there is some open neighbourhood of z in $\tilde{\Sigma}_i$ on which $|H_i^m| < \delta_7$. By recalling the definition of ξ in (3.5.9), and the definition of ξ_i^m that follows (3.5.9), we see that $\xi_i^m \equiv 1$ in this neighbourhood.

Using the definition of $\mathscr{H}_i$ given in (3.5.10), and recalling that $\xi_i^m \equiv 1$ in an open neighbourhood of z, we can compute that having $D\mathscr{H}_i(z)[v] = D\mathscr{H}_i(z)[w]$ for $v, w \in T_z\tilde{\Sigma}_i$ requires $DH_i^m(z)[v] = DH_i^m(z)[w]$. However, H_i^m is a diffeomorphism and thus, in particular, $DH_i^m(z)$ is injective. Therefore we must have that $v = w$ above which establishes that $D\mathscr{H}_i(z)$ is injective. The arbitrariness of $z \in \tilde{\Sigma}_i$ allows us to conclude that for any $z \in \tilde{\Sigma}_i$ the map $D\mathscr{H}_i(z)$ is injective, which in turn establishes that $\mathscr{H}_i$ is an immersion.

Thus we have established that $\mathscr{H}_i$ is a smooth injective immersion on $\tilde{\Sigma}_i$. Therefore, by further shrinking the domain, we can ensure that $\mathscr{H}_i$ is a smooth embedding. To be precise we consider

$$\Omega_i := \bigcup_{m=1}^{\mu} (H_i^m)^{-1}(\mathbb{B}^n(0, \delta_6)) \tag{3.5.16}$$

and note that $\Omega_i \subset\subset \tilde{\Sigma}_i$. To establish that $\mathscr{H}_i$ is a smooth embedding we need only establish that it is a topological embedding, i.e. gives a homeomorphism onto its image. Since $\mathscr{H}_i$ is a smooth injection we may deduce that there is an inverse to $\mathscr{H}_i$ defined on the image $\mathscr{H}_i(\Omega_i)$. We need only establish that this inverse is continuous.

For this purpose suppose that it is not continuous, so we can find a sequence $\{z_j\}_{j=1}^{\infty} \subset \Omega_i$ and a point $z \in \Omega_i$ such that $\mathscr{H}_i(z_j) \to \mathscr{H}_i(z)$ as $j \to \infty$ but $z_j \not\to z$ as $j \to \infty$. After passing to a subsequence in j we may assume that $\{z_j\}_{j=1}^{\infty} \subset \Omega_i \setminus \mathbb{B}_{g_i}(z, \varepsilon)$ for some $\varepsilon > 0$. The compactness of $\overline{\Omega}_i$ allows us to deduce that, after passing to a further subsequence in j, there is a point $y \in \overline{\Omega}_i \setminus \mathbb{B}_{g_i}(x, \varepsilon)$ such that $z_j \to y$ as $j \to \infty$. The continuity of $\mathscr{H}_i$ then ensures that $\mathscr{H}_i(z_j) \to \mathscr{H}_i(y)$ as $j \to \infty$. In turn, this means that $\mathscr{H}_i(y) = \mathscr{H}_i(x)$ but $x \neq y$, which contradicts the fact that $\mathscr{H}_i$ is injective on $\tilde{\Sigma}_i \supset \overline{\Omega}_i$. Thus we must instead have that the inverse $\mathscr{H}_i^{-1}$ is continuous. Hence the map $\mathscr{H}_i : \Omega_i \to \mathbb{R}^N$ is a smooth embedding as claimed and thus $\mathscr{H}_i(\Omega_i)$ is a C^∞ embedded submanifold of $\mathbb{R}^N$.

Observe that the inclusions of (3.5.7) yield that $H_i^m(\mathbb{B}_{g_i}(p_i^m, \delta_1)) \subset \mathbb{B}^n(0, (1 + \eta_0)\delta_1) \subset\subset$

$\mathbb{B}^n(0,\delta_6)$, and so $B_i \subset\subset \Omega_i$. Thus our set of interest B_i is still compactly contained within Ω_i. Our aim now is to use the smooth functions $F_i^{k,m}, f_i^{k,m}$ for $k, m \in \{1,\ldots,\mu\}$ in order to express each embedded submanifold $\mathscr{H}_i(\Omega_i)$ as a union of graphs. First note that if $x \in \mathbb{B}^n(0,\delta_6)$, then $y := (H_i^m)^{-1}(x) \in (H_i^m)^{-1}(\mathbb{B}^n(0,\delta_6))$. Evidently $|H_i^m(y)| \leq \delta_6 < \delta_7$ and so $\xi_i^m(y) = \xi(|x|) = 1$. In turn, this yields that $F_i^{m,m}(x) = H_i^m(y) = x$ and $f_i^{m,m}(x) = 1$.

Now we write $\mathscr{H}_i(\Omega_i)$ as a union of graphs. From (3.5.16) we see that

$$\mathscr{H}_i(\Omega_i) = \bigcup_{m=1}^{\mu} \mathscr{H}_i \circ (H_i^m)^{-1}(\mathbb{B}^n(0,\delta_6)). \tag{3.5.17}$$

For each $m \in \{1,\ldots,\mu\}$ let $t_m : \mathbb{R}^N \to \mathbb{R}^N$ be the coordinate re-ordering function defined by $t_m(x, a_2, \ldots, a_{m-1}, a_{m+1}, \ldots, a_\mu, y_1, \ldots, y_\mu) := (a_2, \ldots, a_{m-1}, x, a_{m+1}, \ldots, a_\mu, y_1, \ldots, y_\mu)$ for given points $x, a_2, \ldots, a_{m-1}, a_{m+1}, \ldots, a_\mu \in \mathbb{R}^n$ and $y_1, \ldots, y_\mu \in \mathbb{R}$. Then we have that

$$\mathscr{H}_i(\Omega_i) = \bigcup_{m=1}^{\mu} \{t_m(x, u_i^m(x)) : x \in \mathbb{B}^n(0,\delta_6)\} \tag{3.5.18}$$

with $u_i^m := (F_i^{1,m}, \ldots, F_i^{m-1,m}, F_i^{m+1,m}, \ldots F_i^{\mu,m}, f_i^{1,m}, \ldots, f_i^{\mu,m})$. Each u_i^m is a smooth function throughout, in particular, $\mathbb{B}^n(0,\delta_7)$, and moreover we have that $u_i^m \to u^m$, where $u^m := (F^{1,m}, \ldots, F^{m-1,m}, F^{m+1,m}, \ldots, F^{\mu,m}, f^{1,m}, \ldots, f^{\mu,m})$, smoothly uniformly, as $i \to \infty$, on $\mathbb{B}^n(0,\delta_7)$. This follows directly from the smooth uniform convergence $F_i^{k,m} \to F^{k,m}$ and $f_i^{k,m} \to f^{k,m}$ as $i \to \infty$ on $\mathbb{B}^n(0,\delta_7)$.

For use later, for each $i \in \mathbb{N}$ and $m \in \{1,\ldots,\mu\}$ we define the map $\psi_{i,m} : \mathbb{B}^n(0,\delta_7) \to \mathbb{R}^N$ by $\psi_{i,m}(x) := t_m(x, u_i^m(x))$ so that

$$\mathscr{H}_i(\Omega_i) = \bigcup_{m=1}^{\mu} \psi_{i,m}(\mathbb{B}^n(0,\delta_6)). \tag{3.5.19}$$

Moreover, if we define $\psi_m : \mathbb{B}^n(0,\delta_7) \to \mathbb{R}^N$ by $\psi_m(x) := t_m(x, u^m(x))$ then we immediately deduce, thanks to the regularity above, that $\psi_{i,m} \to \psi_m$ smoothly uniformly on $\mathbb{B}^n(0,\delta_7)$ as $i \to \infty$. For later use we observe that the uniform (in i) C^l estimates we previously obtained for the functions $F_i^{k,m}$ and $f_i^{k,m}$ over $\mathbb{B}^n(0,\delta_7)$ allow us to conclude uniform C^l estimates, with the same dependencies as specified in (3.5.14), for $\psi_{i,m}$ throughout $\mathbb{B}^n(0,\delta_7)$. Similarly, by considering the uniform C^l estimates for the functions $F^{k,m}$ and $f^{k,m}$ instead, we obtain uniform C^l estimates, with the same dependencies as above, for ψ_m throughout $\mathbb{B}^n(0,\delta_7)$.

For the purposes of writing $\mathscr{H}_i(\Omega_i)$ as a union of graphs we only need the maps $\psi_{i,m}$ on $\mathbb{B}^n(0,\delta_6)$. However later it will be useful to know both that these functions are smooth on the bigger ball $\mathbb{B}^n(0,\delta_7)$ and that we still have uniform (in i) C^l estimates over the bigger ball $\mathbb{B}^n(0,\delta_7)$.

Now consider the subset of $\mathbb{R}^N$ defined by

$$\tilde{M} := \bigcup_{m=1}^{\mu} \{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_6)\} = \bigcup_{m=1}^{\infty} \psi_m\left(\mathbb{B}^n(0, \delta_6)\right), \tag{3.5.20}$$

which is the smooth limit, as $i \to \infty$, of $\mathscr{H}_i(\Omega_i)$. We will establish that $\tilde{M}$ is in fact a C^∞ smooth embedded submanifold.

Before establishing this, we observe that for every $i \in \mathbb{N}$ and each $m \in \{1, \ldots, \mu\}$ the maps $\psi_{i,m}$ and ψ_m are injective immersions. This is immediate from recalling that $\psi_{i,m}(x) = t_m(x, u_i^m(x))$ and $\psi_m(x) = t_m(x, u^m(x))$. From these formulae, we see that both $\psi_{i,m}$ and ψ_m have one component being the identity map $x \mapsto x$. So, by arguing in a similar manner as we did to show that $\mathscr{H}_i$ is an injective immersion, we conclude that $\psi_{i,m}$ and ψ_m are injective immersions as claimed.

A priori, $\tilde{M}$ may be split into several components. But the smooth uniform convergence prevents this. Indeed, consider two points $x, y \in \tilde{M}$. Then for some $m, k \in \{1, \ldots, \mu\}$ we have that $x = \psi_m(a)$ and $y = \psi_k(b)$ for $a, b \in \mathbb{B}^n(0, \delta_6)$. Then x may be connected to $\psi_m(0)$ and y to $\psi_k(0)$ by smooth paths within $\tilde{M}$. Recall that $\psi_m(0)$ is the limit of the images of p_i^m under $\mathscr{H}_i$, and $\psi_k(0)$ is the limit of the images of p_i^k under $\mathscr{H}_i$.

Since $p_i^m, p_i^k \in B_i := \mathbb{B}_{g_i}(x_i, s)$ we are able to choose a smooth path within B_i connecting p_i^m to p_i^k. Since $B_i \subset\subset \Omega_i$ this path lies within Ω_i. The image of this path then gives a smooth path within $\mathscr{H}_i(\Omega_i)$ connecting the image of p_i^m to the image of p_i^k. In the limit $i \to \infty$, the smooth convergence of $\mathscr{H}_i(\Omega_i)$ to $\tilde{M}$ ensures that this path passes to a path within $\tilde{M}$ connecting $\psi_m(0)$ and $\psi_k(0)$. By concatenating the paths x to $\psi_m(0)$, $\psi_m(0)$ to $\psi_k(0)$ and $\psi_k(0)$ to y we obtain a path within $\tilde{M}$ connecting x to y. Since x and y were arbitrarily chosen we may conclude that $\tilde{M}$ is path connected, and thus connected.

We now turn our attention to proving that $\tilde{M}$ is a smooth manifold. To do so, we will show that $\{(\psi_m\left(\mathbb{B}^n(0, \delta_6)\right), \psi_m) : m \in \{1, \ldots, \mu\}\}$ gives a smooth atlas for $\tilde{M}$. The first step is to establish that each $\psi_m\left(\mathbb{B}^n(0, \delta_6)\right)$ for $m \in \{1, \ldots, \mu\}$ is open in $\tilde{M}$ when $\tilde{M}$ is equipped with the subspace topology inherited from $\mathbb{R}^N$. The following claim proves this.

<u>Claim:</u> For $m \in \{1, \ldots, \mu\}$ and $z \in \psi_m(\mathbb{B}^n(0, \delta_6))$ there is a positive radius α, depending on z, such that $\mathbb{B}^N(z, \alpha) \cap \tilde{M} \subset\subset \psi_m(\mathbb{B}^n(0, \delta_6))$.

<u>Proof:</u> It suffices to establish this for the case where $m = 1$. Let $z = (x, u^1(x)) \in \psi_1(\mathbb{B}^n(0, \delta_6)) = \mathbb{B}^n(0, \delta_6) \times u^1(\mathbb{B}^n(0, \delta_6)) \subset \mathbb{R}^N$ and define

$$\alpha := \frac{1}{4} \min\{1, \delta_6 - |x|\} > 0. \tag{3.5.21}$$

Consider $w \in \tilde{M} \cap \mathbb{B}^N(z, \alpha)$. Write $w = (w^1, w^2)$ for $w^1 \in \mathbb{R}^n$ and $w^2 \in \mathbb{R}^{N-n}$. Then we have that $|x - w^1| < \alpha$ and hence (3.5.21) ensures that $w^1 \in \mathbb{B}^n\left(0, \frac{\delta_6+3|x|}{4}\right) \subset\subset \mathbb{B}^n(0,\delta_6)$. To conclude, we need only establish that $w^2 = u^1(w^1)$.

Consider $w_i \in \mathscr{H}_i(\Omega_i)$ such that $w_i \to w$ as $i \to \infty$. If we write $w_i = (w_i^1, w_i^2)$ for $w_i^1 \in \mathbb{R}^n$ and $w_i^2 \in \mathbb{R}^{N-n}$, then this convergence ensures that, for sufficiently large i, we have $|w_i^1 - x| < 2\alpha$. Thus (3.5.21) yields that $w_i^1 \in \mathbb{B}^n(0,\delta_6)$.

For such large i let $\tilde{w}_i := \mathscr{H}_i^{-1}(w_i)$ and note that, since $|x| < \delta_6$, we have that $f_i^{1,1}(x) = 1$. Together with the definition of $\mathscr{H}_i$ in (3.5.10), the definition of u_i^1, and the estimate $|\mathscr{H}_i(\tilde{w}_i) - (x, u_i^1(x))| < 2\alpha$, this gives that $|\xi_i^1(\tilde{w}_i) - 1| < 2\alpha$. Thus, recalling (3.5.21), we have that $\xi_i^1(\tilde{w}_i) > 1 - 2\alpha \geq \frac{1}{2} > 0$. In particular, this tells us that $\tilde{w}_i$ is in the domain of H_i^1 and so $\tilde{w}_i = \left(H_i^1\right)^{-1}(q)$ for some $q \in H_i^1\left(\mathbb{B}_{g_i}(p_i^1, \delta_0)\right)$.

We want to establish that $q = w_i^1$. By recalling the definition of $\mathscr{H}_i$ in (3.5.10), we see that the first n components of $\mathscr{H}_i\left(\left(H_i^1\right)^{-1}(q)\right)$ are given by $\xi(|q|)q$, and so since $\mathscr{H}_i\left(\left(H_i^1\right)^{-1}(q)\right) = \mathscr{H}_i(\tilde{w}_i) = w_i = (w_i^1, w_i^2)$ we have that $w_i^1 = \xi(|q|)q$. If we can show that $\xi(|q|) = 1$ we will obtain our desired equality. If $|q| \leq \delta_6$ then, recalling (3.5.9), we will have $\xi(|q|) = 1$. Note that since $\delta_6 < \delta_0 < 1$, (3.5.21) tells us that $2\alpha \leq \frac{\delta_6 - |x|}{1+\delta_6}$. Therefore, since $\xi(|q|) > 1 - 2\alpha$, we can compute that $|q| = \frac{1}{\xi(|q|)}|w_i^1| < \frac{1}{1-2\alpha}|w_i^1| \leq \frac{1}{1-2\alpha}(|x| + 2\alpha) \leq \delta_6$. Hence $\xi(|q|) = 1$, so $w_i^1 = q$, and $w_i^2 = u_i^1(w_i^1)$. Armed with this knowledge we estimate that

$$|w^2 - u^1(w^1)| \leq |w^2 - w_i^2| + |w_i^2 - u_i^1(w_i^1)| + |u_i^1(w_i^1) - u_i^1(w^1)| + |u_i^1(w^1) - u^1(w^1)|$$

and consider the limit as $i \to \infty$. All four terms on the right hand side converge to 0 as $i \to \infty$, and so $w^2 = u^1(w^1)$, which yields that $w \in \psi_1(\mathbb{B}^n(0,\delta_6))$. Hence we can conclude that $\mathbb{B}^N(z,\alpha) \cap \tilde{M} \subset\subset \psi_1(\mathbb{B}^n(0,\delta_6))$ as claimed. ††

With the claim established, we move on to considering the transition maps $\psi_k^{-1} \circ \psi_m$ for $m, k \in \{1, \ldots, \mu\}$. For this purpose, it is convenient to extend the maps ψ_m by translation in the normal directions. It is here that we make use of the previous observation that the maps ψ_m are smoothly defined throughout $\mathbb{B}^n(0,\delta_7)$, as opposed to $\mathbb{B}^n(0,\delta_6)$, and also satisfy the same uniform C^l estimates throughout the larger ball.

Given $x \in \mathbb{B}^n(0,\delta_7)$, we can write $x = \sum_{k=1}^n x_k e_k$ where $e_1, \ldots, e_n$ denotes the standard basis of $\mathbb{R}^n$. Recall that $\psi_m(x) = t_m(x, u^m(x))$, and the vectors $D_{e_1}\psi_m(x), \ldots, D_{e_n}\psi_m(x) \in \mathbb{R}^N$ span the tangent space to the image of ψ_m at $\psi_m(x)$, viewed as an n-dimensional vector subspace of $\mathbb{R}^N$ passing through $\psi_m(x)$. For each $k \in \{1, \ldots, n\}$ we have

$$D_{e_k}\psi_m(x) = t_m(e_k, D_{e_k}u^m(x)). \tag{3.5.22}$$

If we let $u^m(x) = ((u^m)_1(x), \ldots, (u^m)_{N-n}(x))$ then from (3.5.22) we see that the vectors

$\{v_j(x)\}_{j=1}^{N-n}$ defined by

$$v_j(x) := t_m \left((-D_{e_1}(u^m)_j(x), \ldots, -D_{e_n}(u^m)_j(x)) + e_{n+j} \right), \tag{3.5.23}$$

where here e_{n+j} is the $(n+j)^{\text{th}}$ member of the standard orthonormal basis of $\mathbb{R}^N$, span the orthogonal complement of the tangent space at the point $\psi_m(x)$. For each $x \in \mathbb{B}^n(0, \delta_7)$, these vectors provide the normal directions at each point in the image in which we will translate the image of ψ_m.

We first extend ψ_m to a function $\tilde{\psi}_m : \mathbb{B}^n(0, \delta_7) \times \mathbb{B}^{N-n}(0, 1) \to \mathbb{R}^N$ by defining

$$\tilde{\psi}_m \left(\sum_{k=1}^{N} x_k e_k \right) := \psi_m \left(\sum_{k=1}^{n} x_k e_k \right) + x_{n+1} v_{n+1} \left(\sum_{k=1}^{n} x_k e_k \right) + \ldots + x_N v_N \left(\sum_{k=1}^{n} x_k e_k \right).$$

In terms of the standard basis $e_1, \ldots, e_N \in \mathbb{R}^N$, if we write $\mathbf{x} = \sum_{k=1}^{N} x_k e_k$ and $\mathbf{a} = \sum_{k=1}^{n} x_k e_k$, this is given by

$$\tilde{\psi}_m(\mathbf{x}) = t_m \left(\sum_{k=1}^{n} \left(x_k - \sum_{j=1}^{N-n} x_{n+j} D_{e_k}(u^m)_j(\mathbf{a}) \right) e_k + \sum_{j=1}^{N-n} \left((u^m)_j(\mathbf{a}) + x_{n+j} \right) e_{n+j} \right), \tag{3.5.24}$$

where when a function defined on $\mathbb{B}^n(0, \delta_7)$ is evaluated at the vector $\mathbf{a}$ we mean that it is evaluated at the natural projection of $\mathbf{a}$ to the ball $\mathbb{B}^n(0, \delta_7)$. From (3.5.24) we compute that, up to reordering the columns, the Jacobian matrix of $\tilde{\psi}_m$ is given by $\mathbf{A} + P$ where $\mathbf{A}$ is the $(N \times N)$ matrix

$$\mathbf{A} = \begin{bmatrix} 1 & \ldots & 0 & -D_{e_1}(u^m)_1 & \ldots & -D_{e_1}(u^m)_{N-n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & -D_{e_n}(u^m)_1 & \ldots & -D_{e_n}(u^m)_{N-n} \\ D_{e_1}(u^m)_1 & \ldots & D_{e_n}(u^m)_1 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ D_{e_1}(u^m)_{N-n} & \ldots & D_{e_n}(u^m)_{N-n} & 0 & \ldots & 1 \end{bmatrix} \tag{3.5.25}$$

and P is the $(N \times N)$ matrix

$$P = \sum_{j=1}^{N-n} \begin{bmatrix} x_{n+j} D_{e_1} D_{e_1}(u^m)_j & \ldots & x_{n+j} D_{e_n} D_{e_1}(u^m)_j & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{n+j} D_{e_1} D_{e_n}(u^m)_j & \ldots & x_{n+j} D_{e_n} D_{e_n}(u^m)_j & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{3.5.26}$$

Observe that $\mathbf{A} = I_N + \tilde{\mathbf{A}}$ where I_N denotes the $(N \times N)$ identity matrix and $\tilde{\mathbf{A}}$ is a skew-symmetric $(N \times N)$ matrix, i.e. $\tilde{\mathbf{A}} + \tilde{\mathbf{A}}^T = 0$. All eigenvalues of $\mathbf{A}$ are of the form $1 + b$ where b is an eigenvalue of $\tilde{\mathbf{A}}$. Since $\tilde{\mathbf{A}}$ is skew-symmetric, all its eigenvalues are either 0 or purely imaginary. Further, given a purely imaginary eigenvalue, it follows that the complex conjugate of this eigenvalue is itself an eigenvalue. Since $(1+ip)(1-ip) = 1 + p^2$ for $p \in \mathbb{R}$, and the purely imaginary eigenvalues always occur in complex conjugate pairs, we may compute that

$$\det(\mathbf{A}) = \prod_{b \text{ eigenvalue of } \tilde{\mathbf{A}}} (1+b) = \prod_{\pm ip \text{ eigenvalues of } \tilde{\mathbf{A}}} \left(1 + p^2\right) \geq 1 \tag{3.5.27}$$

at every point in $\mathbb{B}^n(0, \delta_7)$.

Now we turn our attention to the matrix P as defined in (3.5.26). Recall that we have previously shown that the C^2 norm of u^m over the ball $\mathbb{B}^n(0, \delta_7)$ is bounded by a constant depending only on n, C_0, v, R and ρ. Hence from (3.5.26) we can conclude that

$$||P|| := \left(\sum_{a=1}^{N} \sum_{b=1}^{N} |P_{ab}|^2 \right)^{\frac{1}{2}} \leq Q(n, C_0, v, R, \rho) \left(\sum_{j=n+1}^{N} |x_j|^2 \right)^{\frac{1}{2}}. \tag{3.5.28}$$

Since $\det$ is continuous on $M_N(\mathbb{R})$ (the vector space of $(N \times N)$ matrices over $\mathbb{R}$), and since we have observed in (3.5.27) that $\det(\mathbf{A}) \geq 1$, we can conclude that there is a constant $\tau_1 = \tau_1(n, C_0, v, R, \rho) > 0$ such that if $\left(\sum_{j=n+1}^{N} |x_j|^2 \right)^{\frac{1}{2}} \leq \tau_1$ then $|\det(\mathbf{A} + P)| \geq \frac{1}{2}$. Since switching the order of the columns of a matrix only affects the sign of the determinant, we may conclude that $\left|\det\left[D\tilde{\psi}_m\right]\right| \geq \frac{1}{2}$ throughout $\mathbb{B}^n(0, \delta_7) \times \mathbb{B}^{N-n}(0, \tau_1)$.

Given a point $\mathbf{x} \in \mathbb{B}^n(0, \delta_7) \times \mathbb{B}^{N-n}(0, \tau_1)$, we can appeal to the inverse function theorem to deduce that $\tilde{\psi}_m$ is a diffeomorphism onto its image on some neighbourhood of $\mathbf{x}$. The size of this neighbourhood depends on an upper bound on the norm of $D\tilde{\psi}_m$ and an upper bound on the norm of $\left(D\tilde{\psi}_m\right)^{-1}$ at $\tilde{\psi}_m(\mathbf{x})$. Recalling both (3.5.25) and (3.5.26), we see that the C^1 norm of $\tilde{\psi}_m$ over $\mathbb{B}^n(0, \delta_7) \times \mathbb{B}^{N-n}(0, \tau_1)$ is bounded above by a constant which depends only on n, C_0, v, R and ρ. Further, via the formula for matrix inversion and that we have ensured that $\left|\det\left[D\tilde{\psi}_m(\mathbf{x})\right]\right| \geq \frac{1}{2}$, we can conclude that the norm of $\left(D\tilde{\psi}_m\right)^{-1}$ at $\tilde{\psi}_m(\mathbf{x})$ may be bounded above by a constant depending only on n, C_0, v, R and ρ.

Thus, if we restrict to considering the smaller subset $\mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, \tau_1/2)$ of the domain, we may conclude that there is a constant $\nu = \nu(n, C_0, v, R, \rho) > 0$ such that for every $\mathbf{x} \in \mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, \tau_1/2)$ the restriction of $\tilde{\psi}_m$ to $\mathbb{B}^N(\mathbf{x}, \nu)$ is a diffeomorphism onto its image. That is, the neighbourhood of $\mathbf{x}$ upon which the restriction of $\tilde{\psi}_m$ gives a diffeomorphism onto its image can be taken to be a ball of radius $\nu = \nu(n, C_0, v, R, \rho) > 0$.

Now suppose that $\tilde{\psi}_m(\mathbf{x}_1) = \tilde{\psi}_m(\mathbf{x}_2)$ for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, s)$ for some $0 <$

$s \leq \frac{\tau_1}{2}$ to be specified. By using the Fundamental Theorem of Calculus on the components, we can verify that if s is chosen sufficiently small, depending only on n, C_0, v, R and ρ, then we must have that $|\mathbf{x}_1 - \mathbf{x}_2| < \frac{\nu}{2}$, say. But from above we know that $\tilde{\psi}_m$ is a diffeomorphism onto its image once restricted to $\mathbb{B}^N(\mathbf{x}_1, \nu)$, and so we must have that $\mathbf{x}_1 = \mathbf{x}_2$. Thus $\tilde{\psi}_m$ is injective on $\mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, s)$ provided s is sufficiently small, depending only on n, C_0, v, R and ρ.

Combining all our prior observations regarding $\tilde{\psi}_m$, we deduce that there is a constant $\tau = \tau(n, C_0, v, R, \rho) > 0$ such that the restriction of $\tilde{\psi}_m$ to $\mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, \tau)$ is a diffeomorphism onto its image. By repeating for each $m \in \{1, \ldots, \mu\}$ we obtain that all the maps $\tilde{\psi}_m$ are diffeomorphisms onto their images once restricted to $\mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, \tau)$.

Therefore all maps of the form $\tilde{\psi}_m^{-1} \circ \tilde{\psi}_k$ for $m, k \in \{1, \ldots, \mu\}$ are instantly seen to be C^∞ smooth throughout $\tilde{\psi}_k^{-1}\left(\tilde{\psi}_k\left(\mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, \tau)\right) \cap \tilde{\psi}_m\left(\mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, \tau)\right)\right)$. Moreover, since each ψ_m, for $m \in \{1, \ldots, \mu\}$, is the restriction of $\tilde{\psi}_m$ to $\mathbb{B}^n(0, \delta_6) \times \{0\} \subset \mathbb{R}^N$, the maps $\psi_m^{-1} \circ \psi_k$, for all $m, k \in \{1, \ldots, \mu\}$, are the restrictions of the maps $\tilde{\psi}_m^{-1} \circ \tilde{\psi}_k$ to $\tilde{\psi}_k^{-1}\left(\tilde{\psi}_k\left(\mathbb{B}^n(0, \delta_6) \times \{0\}\right) \cap \tilde{\psi}_m\left(\mathbb{B}^n(0, \delta_6) \times \{0\}\right)\right)$.

Observe that for each $m \in \{1, \ldots, \mu\}$ we have that $\tilde{\psi}_m\left(\mathbb{B}^n(0, \delta_6) \times \{0\}\right) = \psi_m\left(\mathbb{B}^n(0, \delta_6)\right)$, and previously we have established in that these sets are open subsets of $\tilde{M}$ with respect to the subspace topology. Hence for every $m, k \in \{1, \ldots, \mu\}$ the subset of $\tilde{M}$ given by the intersection $\tilde{\psi}_k\left(\mathbb{B}^n(0, \delta_6) \times \{0\}\right) \cap \tilde{\psi}_m\left(\mathbb{B}^n(0, \delta_6) \times \{0\}\right)$ is an open with respect to the subspace topology. As such, the maps $\psi_m^{-1} \circ \psi_k$ are the restriction of smooth maps to an open subset with respect to the subspace topology, and thus are themselves C^∞ smooth. Therefore the collection $\{\psi_m\left(\mathbb{B}^n(0, \delta_6)\right), \psi_m\}_{m=1}^{\mu}$ forms a smooth atlas for $\tilde{M}$.

For later convenience we note that the same argument applied to the analogous normal extensions $\tilde{\psi}_{i,m}$ of the maps $\psi_{i,m}$, for $m \in \{1, \ldots, \mu\}$ and $i \in \mathbb{N}$, yields that all maps of the form $\psi_{i,m}^{-1} \circ \psi_{i,k}$ for $m, k \in \{1, \ldots, \mu\}$, are C^∞ smooth and hence the collection $\{\psi_{i,m}\left(\mathbb{B}^n(0, \delta_6)\right), \psi_{i,m}\}_{m=1}^{\mu}$ forms a smooth atlas for $\mathscr{H}_i(\Omega_i)$. It remains to prove that $\tilde{M}$ is embedded. Recall from (3.5.20) that

$$\tilde{M} = \bigcup_{m=1}^{\mu} \psi_m\left(\mathbb{B}^n(0, \delta_6)\right) = \bigcup_{m=1}^{\mu} \{t_m(z, u^m(z)) : z \in \mathbb{B}^n(0, \delta_6)\}. \tag{3.5.29}$$

As we have observed previously, $\psi_m(\mathbb{B}^n(0, \delta_6))$ is the graph of a smooth function it is itself a C^∞ smooth embedded submanifold of $\mathbb{R}^N$. We need to establish that if $V \subset \tilde{M}$ is open, then it is also open in the subspace topology inherited from $\mathbb{R}^N$, i.e. we need to prove that V must contain a set of the form $\mathcal{U} \cap \tilde{M}$ for some open subset $\mathcal{U} \subset \mathbb{R}^N$. It suffices to establish this for $V \subset \psi_m(\mathbb{B}^n(0, \delta_6))$ for some $m \in \{1, \ldots, \mu\}$.

In this case, since $\psi_m(\mathbb{B}^n(0, \delta_6))$ itself is an embedded submanifold, we can conclude that V contains $\mathcal{U} \cap \psi_m(\mathbb{B}^n(0, \delta_6))$ for some open subset $\mathcal{U} \subset \mathbb{R}^N$. If we could choose $\mathcal{U}$ such that it intersected $\tilde{M}$ only within $\psi_m(\mathbb{B}^n(0, \delta_6))$ then we would immediately be able to conclude that $\tilde{M}$

is embedded. A priori, we could have that $\mathcal{U} \cap \tilde{M}$ is a strict superset of $\mathcal{U} \cap \psi_m(\mathbb{B}^n(0, \delta_6))$, and so *not* contained within V.

However, in our previous claim we established that given a point $z \in \psi_m(\mathbb{B}^n(0, \delta_6))$ there is a positive radius α, depending on z, such that $\mathbb{B}^N(z, \alpha) \cap \tilde{M} \subset\subset \psi_m(\mathbb{B}^n(0, \delta_6))$. Replacing $\mathcal{U}$ by $\mathcal{U} \cap \mathbb{B}^N(z, \alpha)$ for some $z \in V$, we see that $\mathcal{U} \cap \tilde{M} = U \cap \psi_m(\mathbb{B}^n(0, \delta_6)) \subset V$, which allows us to conclude that $\tilde{M}$ is a C^∞ smooth embedded submanifold. Therefore there exists a tubular neighbourhood of $\tilde{M}$ in $\mathbb{R}^N$. That is, there exists a neighbourhood Z of the zero section of the normal bundle $N\tilde{M}$ of $\tilde{M}$ in $\mathbb{R}^N$ such that

$$\exp^\perp\big|_Z : Z \to O := \exp^\perp(Z) \subset \mathbb{R}^N$$

is a diffeomorphism onto its image, and $\tilde{M} \subset O$. Let

$$\hat{\Omega}_i := \bigcup_{m=1}^{\mu} (H_i^m)^{-1} (\mathbb{B}^n(0, \delta_5)) \tag{3.5.30}$$

and

$$\hat{M} := \bigcup_{m=1}^{\mu} \{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_5)\} = \bigcup_{m=1}^{\mu} \psi_m(\mathbb{B}^n(0, \delta_5)) \tag{3.5.31}$$

so that $\mathscr{H}_i(\hat{\Omega}_i)$ converges smoothly to $\hat{M}$ as $i \to \infty$. Thus given any $\varepsilon > 0$ we have, for sufficiently large i, both the inclusions $\hat{M} \subset (\mathscr{H}_i(\hat{\Omega}_i))_\varepsilon$ and $\mathscr{H}_i(\hat{\Omega}_i) \subset (\hat{M})_\varepsilon$. Observe that we may consider sufficiently small $\varepsilon > 0$ so that

$$(\hat{M})_\varepsilon \subset \bigcup_{m=1}^{\mu} \tilde{\psi}_m(\mathbb{B}^n(0, \delta_6) \times \mathbb{B}^{N-n}(0, \tau)) \tag{3.5.32}$$

where the maps $\tilde{\psi}_m$ for $m \in \{1, \dots, \mu\}$ are the normal extensions of the maps ψ_m as defined previously. Moreover, $\hat{M} \subset\subset \tilde{M}$, and so for sufficiently small $\sigma > 0$ we have that $(\hat{M})_\sigma \subset O$.

Then we may consider the projection map $\pi : (\hat{M})_\sigma \to \tilde{M}$ projecting $\left(\exp^\perp\big|_Z\right)^{-1}(x)$ to $\tilde{M}$ along normal geodesics. This map satisfies that $\pi(w) = w$ for all $w \in \hat{M}$, and further, given any $a > 0$, by selecting a smaller $\sigma > 0$ if necessary, we may assume that $|\pi(z) - z| < a$ for all $z \in (\hat{M})_\sigma$. For sufficiently large i, the above inclusions yield that $\pi_i := \pi\big|_{\mathscr{H}_i(\hat{\Omega}_i)}$ is a well defined map $\mathscr{H}_i(\hat{\Omega}_i) \to \tilde{M}$. Recalling (3.5.32), we see that locally the map π_i is given by $\psi_m \circ \mathrm{Pr} \circ (\tilde{\psi}_m)^{-1} \circ \left(\exp^\perp\big|_Z\right)^{-1}$ where Pr denotes the natural projection map $\mathbb{R}^N \to \mathbb{R}^n \times \{0\} \subset \mathbb{R}^N$.

This observation, together with noting that $\{\mathbb{B}^n(0, \delta_5), \psi_{i,m}\}_{m=1}^{\mu}$ is a smooth atlas for $\mathscr{H}_i(\hat{\Omega}_i)$ and $\{\mathbb{B}^n(0, \delta_6), \psi_m\}_{m=1}^{\mu}$ is a smooth atlas for $\tilde{M}$, allow us to conclude that the maps π_i are smooth. Further, since $\mathscr{H}_i(\hat{\Omega}_i)$ converges smoothly to $\hat{M}$ as $i \to \infty$, we see that π_i becomes arbitrarily close (in the smooth sense) to the identity map id on $\mathscr{H}_i(\hat{\Omega}_i)$ as $i \to \infty$, and hence must be injective for sufficiently large i. Having established that the inverse π_i^{-1} is well defined, the local form of π_i and the smooth atlases given above allow us to conclude that the inverse is itself smooth. Thus, for sufficiently large i, the map π_i is a diffeomorphism onto its image.

In fact, the smooth closeness to the identity allows us to conclude that for sufficiently large i we have, for any $m \in \{1, \ldots, \mu\}$, that

$$\pi_i\left(\{t_m(z, u_i^m(z)) : z \in \mathbb{B}^n(0, \delta_2)\}\right) \subset\subset \{t_m(z, u^m(z)) : z \in \mathbb{B}^n(0, \delta_3)\} \tag{3.5.33}$$

and

$$\{t_m(z, u^m(z)) : z \in \mathbb{B}^n(0, \delta_4)\} \subset \pi_i\left(\{t_m(z, u_i^m(z)) : z \in \mathbb{B}^n(0, \delta_5)\}\right). \tag{3.5.34}$$

Consider the maps $\phi_i : \hat{\Omega}_i \to \tilde{M}$ given by $\phi_i(x) := \pi_i \circ \mathscr{H}_i(x)$, which are diffeomorphic onto their images. We may observe via (3.5.34) that for sufficiently large i we have

$$\phi_i(\hat{\Omega}_i) = \pi_i\left(\mathscr{H}_i(\hat{\Omega}_i)\right) \supset \tilde{\mathcal{N}} \tag{3.5.35}$$

where $\tilde{\mathcal{N}} := \bigcup_{m=1}^{\mu} \{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_4)\} = \bigcup_{m=1}^{\mu} \psi_m\left(\mathbb{B}^n(0, \delta_4)\right)$. This allows us to consider the smooth maps $\tilde{\varphi}_i : \tilde{\mathcal{N}} \to \mathcal{M}_i$, defined by $\tilde{\varphi}_i := \phi_i^{-1}\big|_{\tilde{\mathcal{N}}}$, which are diffeomorphic onto their image. For convenience we shrink $\tilde{\mathcal{N}}$ further and consider

$$\mathcal{N} := \bigcup_{m=1}^{\mu} \{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_3)\} = \bigcup_{m=1}^{\mu} \psi_m\left(\mathbb{B}^n(0, \delta_3)\right). \tag{3.5.36}$$

For every $i \in \mathbb{N}$ we have $x_i \in (H_i^m)^{-1}\left(\overline{\mathbb{B}^n\left(0, (1+\eta_0)\delta_1\right)}\right)$ for some $m \in \{1, \ldots, \mu\}$, since we know there is some $m \in \{1, \ldots, \mu\}$ for which $d_{g_i}(p_i^m, x_i) \leq \delta_1$. Hence for each $i \in \mathbb{N}$ we have $\mathscr{H}_i(x_i) \in \bigcup_{m=1}^{\mu}\left\{t_m(x, u_i^m(x)) : x \in \overline{\mathbb{B}^n(0, (1+\eta_0)\delta_1)}\right\}$. In particular, the sequence of points $\{\phi_i(x_i) = \pi_i\left(\mathscr{H}_i(x_i)\right)\}_{i=1}^{\infty}$ is contained within $\bigcup_{m=1}^{\mu} \pi\left(\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_2)\}\right)$. Recalling (3.5.33), we see that this is itself compactly contained within $\mathcal{N}$. We can conclude, after potentially passing to a further subsequence in i, that we have $x_0 := \lim_{i\to\infty} \phi_i(x_i) \in \mathcal{N}$. In fact, since every point in B_i is at most a g_i distance δ_1 away from one of the points p_i^m, the above argument allows us to deduce that

$$\phi_i(B_i) = \pi_i\left(\mathscr{H}_i(B_i)\right) \subset \mathcal{V} := \bigcup_{m=1}^{\mu} \pi\left(\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_2)\}\right) \subset\subset \mathcal{N} \tag{3.5.37}$$

for every $i \in \mathbb{N}$. Moreover, the convergence $\phi_i(x_i) \to x_0$ as $i \to \infty$ allows us to pick a sequence of diffeomorphisms $A_i : \mathcal{N} \to \mathcal{N}$, mapping x_0 to $\phi_i(x_i)$, and converging smoothly uniformly to the identity map $\mathrm{id} : \mathcal{N} \to \mathcal{N}$.

The inclusions in (3.5.35) tell us that $\tilde{\varphi}_i(\tilde{\mathcal{N}}) \subset \hat{\Omega}_i$ and so we can first consider the sequence of pulled back metrics $\tilde{\varphi}_i^* g_i$ on $\tilde{\mathcal{N}}$. Consider the pull backs $\left(\mathscr{H}_i^{-1}\right)^* g_i$ of the metrics g_i to $\mathscr{H}_i(\Omega_i)$. The estimates of (3.5.4) in Lemma 3.5.1 tell us that, for each $m \in \{1, \ldots, \mu\}$, the component functions $\left(\left(\mathscr{H}_i^{-1}\right)^* g_i\right)_{\alpha\beta}$ for $\alpha, \beta \in \{1, \ldots, n\}$ satisfy uniform (in i) C^l estimates throughout $\mathscr{H}_i \circ (H_i^m)^{-1} (\mathbb{B}^n(0, \delta_6))$, say. Moreover, for $i \in \mathbb{N}$, the inclusion (3.5.34) tells us that, for each $m \in \{1, \ldots, \mu\}$, we have

$$\pi_i^{-1} \left(\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_4)\}\right) \subset \{t_m(z, u_i^m(z)) : z \in \mathbb{B}^n(0, \delta_5)\}. \tag{3.5.38}$$

Since $\mathscr{H}_i \circ (H_i^m)^{-1} (\mathbb{B}^n(0, \delta_5)) = \{t_m(z, u_i^m(z)) : z \in \mathbb{B}^n(0, \delta_5)\}$, and $\delta_5 < \delta_6$, we see that the component functions $\left(\left(\mathscr{H}_i^{-1}\right)^* g_i\right)_{\alpha\beta}$ for $\alpha, \beta \in \{1, \ldots, n\}$ satisfy uniform (in i) C^l estimates throughout $\pi_i^{-1} \left(\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_4)\}\right)$. By recalling that $\tilde{\varphi}_i := \mathscr{H}_i^{-1} \circ \pi_i^{-1}$ where defined, we deduce that the component functions $(\tilde{\varphi}_i^* g_i)_{\alpha\beta}$ enjoy uniform (in i) C^l estimates throughout $\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_4)\}$. Here we have used that π_i becomes arbitrarily close in the smooth sense to the identity map as $i \to \infty$ to control all C^l norms of π_i and π_i^{-1}, for $l \in \mathbb{N}$, independently of i, for sufficiently large i.

By appealing to Ascoli-Arzelà and passing to a subsequence in i, we see that $\tilde{\varphi}_i^* g_i$ converges smoothly locally to a smooth metric on $\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_4)\}$. Repeating for each $m \in \{1, \ldots, \mu\}$, passing to successive subsequences in i, we may conclude such convergence on all $\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_4)\}$ simultaneously. Since the metrics g_i agree on the overlaps between $(H_i^m)^{-1} (\mathbb{B}^n(0, \delta_6))$, the smooth limits obtained must agree with each other on the regions where the sets $\{t_m(x, u^m(x)) : x \in \mathbb{B}^n(0, \delta_4)\}$ overlap. Hence we get a smooth Riemannian metric g_∞ on $\tilde{\mathcal{N}}$ such that $\tilde{\varphi}_i^* g_i \to g_\infty$ smoothly-locally as $i \to \infty$.

Since $\mathcal{N} \subset\subset \tilde{\mathcal{N}}$ we obtain smooth uniform convergence $\tilde{\varphi}_i^* g_i \to g_\infty$ on $\mathcal{N}$. The uniform smooth convergence $A_i \to \mathrm{id}$ on $\mathcal{N}$ yields that $A_i^* g_\infty \to g_\infty$ smoothly uniformly on $\mathcal{N}$, and so $A_i^* \tilde{\varphi}_i^* g_i \to g_\infty$ smoothly uniformly on $\mathcal{N}$. We can now define our sequence of smooth maps $\varphi_i : \mathcal{N} \to \mathcal{M}_i$ by $\varphi_i := \tilde{\varphi}_i \circ A_i$. As required φ_i maps x_0 to x_i, is a diffeomorphism onto its image, and the sequence satisfies that $\varphi_i^* g_i \to g_\infty$ smoothly uniformly on $\mathcal{N}$ as $i \to \infty$.

Finally, from (3.5.37) we have that for every $i \in \mathbb{N}$ we have $\mathbb{B}_{\varphi_i^* g_i}(x_0, s) = \varphi_i^{-1}(B_i) \subset A_i^{-1}(\mathcal{V}) \subset\subset \mathcal{N}$. Since $A_i \to \mathrm{id}$ smoothly uniformly as $i \to \infty$, then for sufficiently large i, we have $A_i^{-1}(\mathcal{V}) \subset \mathcal{W} \subset\subset \mathcal{N}$ for some fixed open subset $\mathcal{W} \subset\subset \mathcal{N}$. Since $\overline{\mathcal{W}} \subset \mathcal{N}$ is compact, we have, thanks to the smooth uniform convergence $\varphi_i^* g_i \to g_\infty$ on $\mathcal{N}$ as $i \to \infty$, that the metrics g_∞ and $\varphi_i^* g_i$ are uniformly equivalent on $\overline{\mathcal{W}}$, for sufficiently large i. As a consequence, if we let $Q := r + \frac{\rho}{2} \in (r, s)$, then we have that $\mathbb{B}_{g_\infty}(x_0, Q) \subset \mathbb{B}_{\varphi_i^* g_i}(x_0, s) \subset\subset \mathcal{N}$ for sufficiently large i. Since $Q > r$ we can deduce both that $\varphi_i^* g_i \to g_\infty$ on $\overline{\mathbb{B}_{g_\infty}(x_0, r)}$ as $i \to \infty$, along with $\mathbb{B}_{g_\infty}(x_0, r) \subset\subset \mathcal{N}$, which completes the proof of Lemma 3.4.1 ■

3.6. Local Ricci Flow Compactness

This section is taken from [MT18], which is joint work with and Peter M. Topping. The local version of Hamilton-Cheeger-Gromov compactness for flows, which is already implicit in [ST17], is the following.

Theorem 3.6.1 (Local Ricci flow compactness; Lemma B.3 in [MT18])**.** *Suppose $(\mathcal{M}_i^n, g_i(t))$ is a sequence of smooth n-dimensional Ricci flows, not necessarily complete, each defined for $t \in [0, T]$, and with $x_i \in \mathcal{M}_i$ for each $i \in \mathbb{N}$. Suppose that, for some $R > 0$, we have $\mathbb{B}_{g_i(0)}(x_i, R) \subset\subset \mathcal{M}_i$ and $\mathrm{Vol}\mathbb{B}_{g_i(0)}(x_i, R) \geq v > 0$ for each i, and throughout $\mathbb{B}_{g_i(0)}(x_i, R)$ that $\mathrm{Ric}_{g_i(t)} \geq -\alpha < 0$ for all $t \in [0, T]$ and $|\mathrm{Rm}|_{g_i(t)} \leq c_0/t$ for all $t \in (0, T]$, for positive constants v, α and c_0 that are independent of i.*

Then for all $\eta \in (0, R/2)$, there exists $S > 0$ depending only on R, n, v, α, c_0 and η such that after passing to an appropriate subsequence in i, there exist a smooth n-dimensional manifold $\mathcal{N}$, a point $x_0 \in \mathcal{N}$ and a Ricci flow $g(t)$ on $\mathcal{N}$ for $t \in (0, \tau]$, where $\tau := \min\{T, S\}$, with the following properties.

First, $\mathbb{B}_{g(t)}(x_0, R - \eta) \subset\subset \mathcal{N}$ for all $t \in (0, \tau]$. Second, if we define Ω to be the connected component of the interior of

$$\bigcap_{s \in (0,\tau]} \mathbb{B}_{g(s)}(x_0, R - \eta) \subset \mathcal{N}$$

containing x_0, then for all $t \in (0, \tau]$ we have $\mathbb{B}_{g(t)}(x_0, R - 2\eta) \subset \Omega$. Third, there exists a sequence of smooth maps $\varphi_i : \Omega \to \mathbb{B}_{g_i(0)}(x_i, R) \subset \mathcal{M}_i$, diffeomorphic onto their images and mapping x_0 to x_i, such that $\varphi_i^* g_i(t) \to g(t)$ smoothly uniformly on $\Omega \times [\delta, \tau]$ for every $\delta \in (0, \tau)$.

Finally, throughout Ω we have $\mathrm{Ric}_{g(t)} \geq -\alpha$ and $|\mathrm{Rm}|_{g(t)} \leq c_0/t$ for all $t \in (0, \tau]$.

Proof. We begin by applying the *local Shi decay lemma* 2.5.4 to each $g_i(t)$, with $\varepsilon := \eta/3$. This ensures that there exists $S > 0$ depending only on n, c_0 and η such that for $0 < t \leq \min\{T, S\}$ we have $\mathbb{B}_{g_i(t)}(x_i, R - \varepsilon) \subset \mathbb{B}_{g_i(0)}(x_i, R)$ and that for $0 < t \leq \tau \leq \min\{T, S\}$ we have

$$\left|\nabla^l \mathrm{Rm}\right|_{g_i(t)} \leq \frac{C_l}{t^{1+\frac{l}{2}}} \tag{3.6.1}$$

throughout $\mathbb{B}_{g_i(\tau)}(x_i, R - \varepsilon)$, where C_l depends on l, c_0, n and η. Next with a view to later applying the expanding and shrinking balls lemmas, we reduce $S > 0$ further, depending now also on α, so that

$$R(1 - e^{-\alpha S}) < \varepsilon \qquad \text{and} \qquad S \leq \frac{\eta^2}{\beta^2 c_0}. \tag{3.6.2}$$

where $\beta = \beta(n) \geq 1$ comes from Lemma 2.4.6. A final reduction of $S > 0$, depending now also on v, ensures that by Lemma 2.4.9 (which is in a more appropriate form than the variant Lemma 2.4.10) and Bishop-Gromov (Theorem 2.2.1), we have $\mathrm{Vol}\mathbb{B}_{g_i(s)}(x_i, R-\varepsilon) \geq \tilde{v} > 0$ for all $s \in [0, \min\{T, S\}]$, where $\tilde{v}$ depends only on v, α, R and n.

At this point we fix S, and the corresponding $\tau := \min\{T, S\}$, and apply the local compactness Theorem 3.2.1 to the sequence $g_i(\tau)$ with R there equal to $R - \varepsilon$ here. The conclusion is that after passing to an appropriate subsequence in i, there exist a smooth n-dimensional Riemannian manifold $(\mathcal{N}, g_\infty)$, a point $x_0 \in \mathcal{N}$ with $\mathbb{B}_{g_\infty}(x_0, r) \subset\subset \mathcal{N}$ for every $r \in (0, R-\varepsilon)$, and a sequence of smooth maps $\varphi_i : \mathbb{B}_{g_\infty}(x_0, \frac{i}{i+1}(R-\varepsilon)) \to \mathcal{M}_i$, diffeomorphic onto their images and mapping x_0 to x_i, such that $\varphi_i^* g_i(\tau) \to g_\infty$ smoothly locally on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon)$.

By Part 1 of Lemma 3.3.3, applied with g_i and $\hat{g}$ there equal to $g_i(\tau)$ and g_∞ here, respectively, and with $a = 2r$ and b there equal to $R - 2\varepsilon$ and $R - \varepsilon$ here, respectively, we find that after dropping finitely many terms, we have

$$\varphi_i\left(\mathbb{B}_{g_\infty}(x_0, R-2\varepsilon)\right) \subset \mathbb{B}_{g_i(\tau)}\left(x_i, R-\varepsilon\right) \subset \mathbb{B}_{g_i(0)}\left(x_i, R\right)$$

for every i (where the second inclusion here was established at the beginning of the proof).

By Hamilton's original argument [Ham95] we can pass to a further subsequence and find a Ricci flow $g(t)$ on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon)$, $t \in (0, \tau]$ so that $g(\tau) = g_\infty$ on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon)$ and so that $\varphi_i^* g_i(t) \to g_\infty$ smoothly locally on $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon) \times (0, \tau]$ as $i \to \infty$. In particular, we can pass our curvature hypotheses to the limit to obtain that $\mathrm{Ric}_{g(t)} \geq -\alpha$ and $|\mathrm{Rm}|_{g(t)} \leq c_0/t$, for all $t \in (0, \tau]$ and throughout $\mathbb{B}_{g_\infty}(x_0, R-2\varepsilon)$.

Next, our constraint (3.6.2) implies that $(R-2\varepsilon)(1-e^{-\alpha S}) < \varepsilon$, i.e. that $R - 3\varepsilon < (R-2\varepsilon)e^{-\alpha S}$, and hence by the expanding balls lemma 2.4.7, we know that for all $t \in (0, \tau]$ we have

$$\begin{aligned}\mathcal{N} \supset\supset \mathbb{B}_{g(\tau)}(x_0, R-2\varepsilon) &\supset \mathbb{B}_{g(t)}\left(x_0, (R-2\varepsilon)e^{\alpha(t-\tau)}\right) \\ &\supset \mathbb{B}_{g(t)}\left(x_0, (R-2\varepsilon)e^{-\alpha S}\right) \\ &\supset \mathbb{B}_{g(t)}(x_0, R-3\varepsilon),\end{aligned}$$

and hence (recalling that $\varepsilon = \eta/3$) we have $\mathbb{B}_{g(t)}(x_0, R-\eta) \subset \mathbb{B}_{g(\tau)}(x_0, R-2\varepsilon) \subset\subset \mathcal{N}$ as required. One consequence is that our curvature bounds hold within each $\mathbb{B}_{g(t)}(x_0, R-\eta)$, for all $t \in (0, \tau]$. Moreover, if we reduce $\mathcal{N}$ to $\mathbb{B}_{g_\infty}(x_0, R-\varepsilon) \subset \mathcal{N}$, then we still have $\mathbb{B}_{g(t)}(x_0, R-\eta) \subset\subset \mathcal{N}$ for all $t \in (0, \tau]$, and now the Ricci flow is defined throughout $\mathcal{N}$.

It remains to show that $\mathbb{B}_{g(t)}(x_0, R-2\eta) \subset \Omega$, and for that it suffices to prove that

$$\mathbb{B}_{g(t)}(x_0, R-2\eta) \subset \mathbb{B}_{g(s)}(x_0, R-\eta) \qquad \text{for each } s, t \in (0, \tau]. \tag{3.6.3}$$

In the case $s < t$ this follows from the shrinking balls lemma 2.4.6 applied with time zero there equal to time s here, and r there equal to $R-\eta$ here. That lemma tells us that $\mathbb{B}_{g(t)}(x_0, R-\eta-\beta\sqrt{c_0(t-s)}) \subset \mathbb{B}_{g(s)}(x_0, R-\eta)$, by (2.4.8) not (2.4.9), and because $\beta\sqrt{c_0(t-s)} \leq \beta\sqrt{c_0 S} \leq \eta$, by (3.6.2), this implies (3.6.3).

Meanwhile, in the case $s > t$, (3.6.3) follows from the expanding balls lemma 2.4.7 applied with time zero there equal to time s here, and R there equal to $R-\eta$ here. That lemma tells us that $\mathbb{B}_{g(s)}(x_0, R-\eta) \supset \mathbb{B}_{g(t)}(x_0, (R-\eta)e^{\alpha(t-s)})$, and so we will have proved (3.6.3) if we can prove that $(R-\eta)e^{\alpha(t-s)} \geq R-2\eta$, or more generally that $(R-\eta)e^{-\alpha S} \geq R-2\eta$, which is equivalent to $(R-\eta)(1-e^{-\alpha S}) \leq \eta$. This in turn follows from the first part of (3.6.2). ∎

Chapter 4

Global Regularity of 3D Ricci Limit Spaces

This Chapter is taken from [MT18], and is joint work with Peter M. Topping.

4.1. Introduction

Given a sequence of n-dimensional complete, smooth, pointed Riemannian manifolds $(\mathcal{M}_i, g_i, x_i)$, for which $\mathrm{Ric}_{g_i} \geq -\alpha_0$ for some given α_0, Gromov's compactness theorem 2.8.3 tells us that, after passing to a subsequence, there exists a locally compact complete pointed metric space (X, d_X, x_0) for which $(\mathcal{M}_i, d_{g_i}, x_i) \to (X, d_X, x_0)$ in the pointed Gromov-Hausdorff sense; as outlined in Definition 2.8.2. It is a natural question to ask about the regularity of the limit space (X, d_X), continuing a long tradition of such results that originates with the study of limit spaces of manifolds with uniform lower sectional curvature bounds (see e.g. [BBI01] as a starting point). In this chapter we consider the weakly noncollapsed setting, that is with the added assumption that $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$. We refer to this setting as *weakly* noncollapsed since we only require a *single* unit ball $\mathbb{B}_{g_i}(x_i, 1)$ to have a specified uniform lower volume bound as opposed to the stronger globally noncollapsed condition in which we require *all* balls $\mathbb{B}_{g_i}(x, 1)$ to have the uniform lower volume bound. This stronger globally noncollapsed hypothesis can be handled using Ricci flow techniques that are far simpler than those required in this chapter.

Pioneering regularity results were obtained for the limit spaces (X, d_X, x_0) of sequences of n-dimensional manifolds with uniform lower Ricci bounds by Cheeger-Colding, see [Che01], as we now describe. In the weakly noncollapsed setting the *'regular set'* $\mathcal{R}$ of X is the set of points in X at which all tangent cones are n-dimensional Euclidean space; see [Che01]. Cheeger-Colding [CC97] proved that while the Hausdorff dimension of X is n, the singular set $\mathcal{S} := X \setminus \mathcal{R}$

has Hausdorff dimension no larger than $n-2$, and the regular set is contained within an open set that is locally bi-Hölder homeomorphic to a smooth manifold.

Recently, Miles Simon and Peter Topping obtained improved regularity in dimension three; in [ST17] it is proved that weakly noncollapsed Ricci limit spaces in dimension three are topological manifolds throughout the entire limit space, irrespective of singularities. In fact, given any point $x \in X$, including any singular point, there is a neighbourhood of x that is *bi-Hölder* homeomorphic to a ball in $\mathbb{R}^3$. Moreover, the theory in that paper establishes that for any $r > 0$, the ball $\mathbb{B}_{d_X}(x_0, r)$ is bi-Hölder homeomorphic to an open subset in a complete smooth Riemannian manifold. See Theorem 1.4 and Corollary 1.5 in [ST17] for full details. We use all the technology from [ST17] and key results and ideas from [ST16, Hoc16] in order to prove directly the following result.

Theorem 4.1.1 (***Ricci limit spaces are globally smooth manifolds; Theorem 1.1 in [MT18]***)**.** *Suppose that* $\left(\mathcal{M}_i^3, g_i, x_i\right)$ *is a sequence of complete, smooth, pointed Riemannian three-manifolds such that for some* $\alpha_0 > 0$ *and* $v_0 > 0$*, and for all* $i \in \mathbb{N}$*, we have* $\mathrm{Ric}_{g_i} \geq -\alpha_0$ *throughout* $\mathcal{M}_i$*, and* $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$.

Then there exist a smooth three-manifold M*, a point* $x_0 \in M$*, and a complete distance metric* $d : M \times M \to [0, \infty)$ *generating the same topology as* M *such that after passing to a subsequence in* i *we have*

$$\left(\mathcal{M}_i^3, d_{g_i}, x_i\right) \to (M, d, x_0),$$

in the pointed Gromov-Hausdorff sense, and if g *is any smooth complete Riemannian metric on* M *then the identity map* $(M, d) \to (M, d_g)$ *is locally bi-Hölder.*

A key part of [ST17] is the use of Ricci flow to *'mollify'* the Riemannian manifolds $(\mathcal{M}_i, g_i)$ in the spirit of early work of Simon e.g. [Sim02, Sim12]. However, it is not expected that there exists any traditional smooth Ricci flow that starts from a general limit space (X, d_X), or even from a general *smooth* three-manifold with Ricci curvature bounded below [Top14], so in [ST17] a notion of local Ricci flow is used, which automatically generates not just a Ricci flow, but also the underlying smooth manifold for the flow, see e.g. Theorem 1.1 in [ST17]. This Ricci flow is posed within a class of flows with good estimates, and it is not reasonable to ask for uniqueness of solutions. A consequence of this is that if one takes a second local Ricci flow on a larger local region of the limit space, then restricts to the original local region, there is no guarantee that the natural identification of the two resulting smooth underlying manifolds will be smooth. Consequently, Theorem 4.1.1 does not immediately follow.

These considerations encourage us to look again at the idea of trying to imagine a Ricci flow starting from the entire Ricci limit space (X, d_X). We have already pointed out that this should be impossible in the traditional manner, but it is instructive to imagine why we cannot construct such a Ricci flow as a limit of local Ricci flows that exist on larger and larger balls $\mathbb{B}_{g_i}(x_i, i)$. The problem is that the degree of noncollapsing of such balls typically degenerates as $i \to \infty$, and therefore the existence time of the corresponding local Ricci flows may degenerate to zero.

The solution to these problems, refining an approach of Hochard [Hoc16], is to consider Ricci flows that live on a subset of spacetime that is not simply a parabolic cylinder $\mathcal{M} \times [0, T]$. Given a smooth, complete Riemannian three-manifold $(\mathcal{M}, g_0, x_0)$ satisfying the above Ricci lower bound and weakly noncollapsed condition, then for any $k \in \mathbb{N}$, we prove the existence of a smooth Ricci flow $g_k(t)$ that is defined on a subset of spacetime that contains, for each $m \in \{1, \ldots, k\}$, the cylinder $\mathbb{B}_{g_0}(x_0, m) \times [0, T_m]$, where crucially $T_m > 0$ depends only on α_0, v_0 and m, and in particular *not* on k. Further, the flow enjoys local curvature bounds on the set $\mathbb{B}_{g_0}(x_0, m) \times (0, T_m]$, which again depend only on α_0, v_0 and m.

Theorem 4.1.2 (***Pyramid Ricci flow construction; Theorem 1.2 in [MT18]***)**.** *Suppose that* (M^3, g_0) *is a complete smooth Riemannian three-manifold and fix* $x_0 \in M$. *For given* $\alpha_0, v_0 > 0$, *assume we have both* $\mathrm{Ric}_{g_0} \geq -\alpha_0$ *throughout* M, *and* $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0 > 0$.

Then there exist increasing sequences $C_k \geq 1$ *and* $\alpha_k > 0$, *and a decreasing sequence* $T_k > 0$, *all defined for* $k \in \mathbb{N}$, *and depending only on* α_0 *and* v_0, *such that the following is true.*

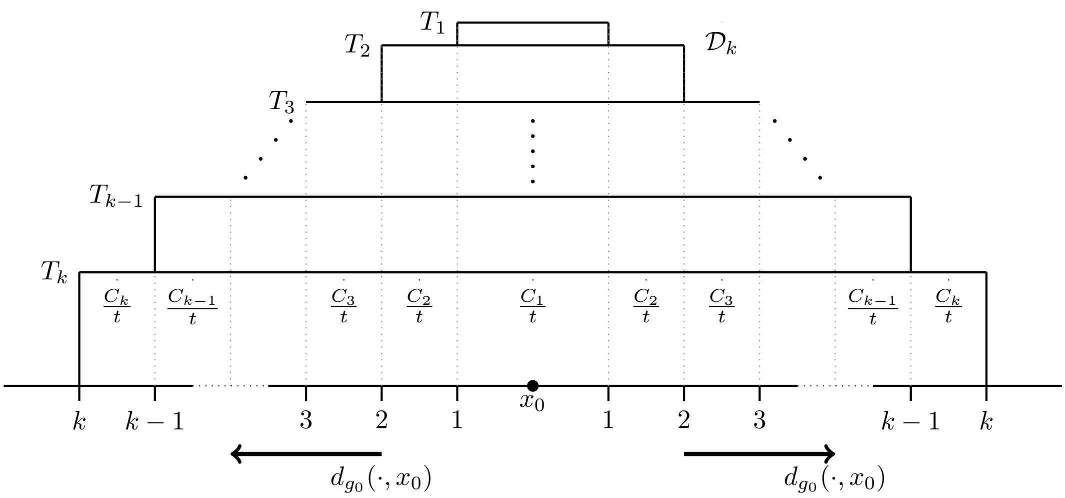
For any $k \in \mathbb{N}$ *there exists a smooth Ricci flow solution* $g_k(t)$, *defined on a subset* $\mathcal{D}_k$ *of spacetime given by*

$$\mathcal{D}_k := \bigcup_{m=1}^{k} \mathbb{B}_{g_0}(x_0, m) \times [0, T_m],$$

with $g_k(0) = g_0$ *on* $\mathbb{B}_{g_0}(x_0, k)$, *and satisfying, for each* $m \in \{1, \ldots, k\}$,

$$\begin{cases} \mathrm{Ric}_{g_k(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_0}(x_0, m) \times [0, T_m] \\ |\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_0}(x_0, m) \times (0, T_m]. \end{cases} \tag{4.1.1}$$

The domain of definition $\mathcal{D}_k$ of the Ricci flow $g_k(t)$ has a pyramid structure, as illustrated in the following figure, and throughout this chapter we term such Ricci flows as *'Pyramid Ricci flows.'*



As the distance from the central point x_0 increases, not only does the existence time of the flow decrease, but the C/t curvature decay estimate worsens. This is in contrast to the *partial* Ricci flow construction of Hochard, and is essential to obtain the uniform estimates on the domain of existence. Another distinction to partial Ricci flows is that by virtue of the theory of Miles Simon and Peter Topping in [ST16, ST17], in particular the so-called Double Bootstrap lemma, our flows have lower Ricci bounds that do not degenerate as $t \downarrow 0$. These uniform lower Ricci bounds will be crucial for obtaining our bi-Hölder estimates in Theorem 4.1.1, and to make the application to Ricci limit spaces, thanks to the bi-Hölder regularity from Lemma 3.1 in [ST17] (see Lemma 2.4.11).

Our pyramid Ricci flows constructed in Theorem 4.1.2 allow us to prove the following hybrid of the local and global existence results from [ST17].

Theorem 4.1.3 (***Global-Local Ricci flows; Theorem 1.3 in [MT18]***). *Suppose that (M, g_0, x_0) is a complete, smooth, pointed, Riemannian three-manifold and, for given $\alpha_0, v_0 > 0$, we have both $\mathrm{Ric}_{g_0} \geq -\alpha_0$ throughout M, and $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0 > 0$. Then there exist increasing sequences $C_j \geq 1$ and $\alpha_j > 0$ and a decreasing sequence $T_j > 0$, all defined for $j \in \mathbb{N}$, and depending only on α_0 and v_0, for which the following is true.*

There exists a smooth Ricci flow $g(t)$, defined on a subset of spacetime that contains, for each $j \in \mathbb{N}$, the cylinder $\mathbb{B}_{g_0}(x_0, j) \times [0, T_j]$, satisfying that $g(0) = g_0$ throughout M, and further that, again for each $j \in \mathbb{N}$,

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_j & \text{on } \mathbb{B}_{g_0}(x_0, j) \times [0, T_j] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_j}{t} & \text{on } \mathbb{B}_{g_0}(x_0, j) \times (0, T_j]\,. \end{cases} \tag{4.1.2}$$

To reiterate, in this result we only assume weak noncollapsing, and thus we must not expect global existence for positive times.

Analogously to Theorem 1.8 from [ST17], we can obtain this sort of global-local existence starting also from a *weakly* noncollapsed Ricci limit space, and in doing so we establish most of Theorem 4.1.1.

Theorem 4.1.4. (***Ricci flow from a weakly noncollapsed 3D Ricci limit space; Theorem 1.4 in [MT18]***) *Suppose that* $\left(\mathcal{M}_i^3, g_i, x_i\right)$ *is a sequence of complete, smooth, pointed Riemannian three-manifolds such that for given* $\alpha_0, v_0 > 0$ *we have* $\mathrm{Ric}_{g_i} \geq -\alpha_0$ *throughout* $\mathcal{M}_i$*, and* $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$*, for each* $i \in \mathbb{N}$*.*

Then there exist increasing sequences $C_k \geq 1$ *and* $\alpha_k > 0$ *and a decreasing sequence* $T_k > 0$*, all defined for* $k \in \mathbb{N}$*, and depending only on* α_0 *and* v_0*, for which the following holds.*

There exist a smooth three-manifold M*, a point* $x_0 \in M$*, a complete distance metric* $d : M \times M \to [0, \infty)$ *generating the same topology as we already have on* M*, and a smooth Ricci flow* $g(t)$ *defined on a subset of spacetime* $M \times (0, \infty)$ *that contains* $\mathbb{B}_d(x_0, k) \times (0, T_k]$ *for each* $k \in \mathbb{N}$*, with* $d_{g(t)} \to d$ *locally uniformly on* M *as* $t \downarrow 0$*, and after passing to a subsequence in* i *we have that* $(\mathcal{M}_i, d_{g_i}, x_i)$ *converges in the pointed Gromov-Hausdorff sense to* (M, d, x_0)*. Moreover, for any* $k \in \mathbb{N}$*,*

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_k & \text{on } \mathbb{B}_d(x_0, k) \times (0, T_k] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_k}{t} & \text{on } \mathbb{B}_d(x_0, k) \times (0, T_k]\,. \end{cases} \tag{4.1.3}$$

This theorem will be a special case of the more elaborate Theorem 4.5.1 that will explicitly arrive at $g(t)$ as a limit of pyramid Ricci flows via pull-back by diffeomorphisms generated by the local form of Hamilton-Cheeger-Gromov compactness given in Theorem 3.6.1. A further special case of Theorem 4.5.1 will be Theorem 4.1.1, and the following stronger assertion.

Theorem 4.1.5 (***Regular GH approximations; Theorem 1.5 in [MT18]***)**.** *In the setting of Theorem 4.1.1, we may assume the following additional conclusions:*

There exists a sequence of smooth maps $\varphi_i : \mathbb{B}_d(x_0, i) \to \mathcal{M}_i$*, diffeomorphic onto their images, and mapping* x_0 *to* x_i *such that for any* $R > 0$ *we have* $d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d(x, y)$ *uniformly for* $x, y \in \mathbb{B}_d(x_0, R)$ *as* $i \to \infty$*.*

Moreover, for sufficiently large i*,* $\varphi_i|_{\mathbb{B}_d(x_0,R)}$ *is bi-Hölder with Hölder exponent depending only on* α_0*,* v_0 *and* R*.*

Finally, for any $r \in (0, R)$*, and for sufficiently large* i*,* $\varphi_i|_{\mathbb{B}_d(x_0,R)}$ *maps onto* $\mathbb{B}_{g_i}(x_i, r)$*.*

Thus, not only do we have the pointed Gromov-Hausdorff convergence of Theorem 4.1.1, we can also find Gromov-Hausdorff approximations (cf. Definition 2.8.1) that are smooth and bi-Hölder (neglecting a thin boundary layer) cf. Theorem 1.4 from [ST17].

Within this Chapter there are several substantial deviations from existing theory. The main novelty is the new pyramid extension lemma 4.2.1 in Section 4.2. This result asserts that it is not just possible to construct a local Ricci flow with good estimates, but that we can additionally assume that this local flow extends a given Ricci flow dened for a shorter time on a larger domain. The estimates, and their constants, are handled with sufficient care that the pyramid extension lemma can be iterated, in Section 4.3, to construct the pyramid Ricci flows of Theorem 4.1.2. Working on a fixed manifold, we use these pyramid Ricci flows to prove Theorem 4.1.3 in Section 4.4. Another notable difference compared to the existing theory arises in the Ricci flow compactness of Section 4.5. For compactness of pyramid flows we must appeal to compactness of the flows not at one time slice, as in the traditional theory, but at countably many time slices. The resulting Theorem 4.5.1 in turn establishes Theorems 4.1.1, 4.1.4 and 4.1.5.

4.2. The Pyramid Extension Lemma

The following result interpolates between the local existence theorem (Theorem 1.6) and the extension lemma (Lemma 4.4) of Simon-Topping [ST17], and is the major ingredient in constructing pyramid Ricci flows.

Lemma 4.2.1 (***Pyramid Extension Lemma; Lemma 2.1 in [MT18]***). *Suppose* (M, g_0, x_0) *is a pointed complete Riemannian 3-manifold satisfying* $\mathrm{Ric}_{g_0} \geq -\alpha_0 < 0$ *throughout* M*, and* $\mathrm{Vol}\mathbb{B}_{g_0}(x_0, 1) \geq v_0 > 0$*. Then there exist increasing sequences* $C_k \geq 1$ *and* $\alpha_k > 0$*, and a decreasing sequence* $S_k > 0$*, all defined for* $k \in \mathbb{N}$ *and depending only on* α_0 *and* v_0*, with the following properties.*

First, for each $k \in \mathbb{N}$ *there exists a Ricci flow* $g(t)$ *on* $\mathbb{B}_{g_0}(x_0, k)$ *for* $t \in [0, S_k]$ *such that* $g(0) = g_0$ *where defined and so that* $|\mathrm{Rm}|_{g(t)} \leq C_k/t$ *for all* $t \in (0, S_k]$ *and* $\mathrm{Ric}_{g(t)} \geq -\alpha_k$ *for all* $t \in [0, S_k]$*.*

Moreover, given any Ricci flow $\tilde{g}(t)$ *on* $\mathbb{B}_{g_0}(x_0, k+1)$ *over a time interval* $t \in [0, S]$ *with* $\tilde{g}(0) = g_0$ *where defined, and with* $|\mathrm{Rm}|_{\tilde{g}(t)} \leq c_0/t$ *for some* $c_0 > 0$ *and all* $t \in (0, S]$*, there exists* $\tilde{S}_k > 0$ *depending on* k*,* α_0*,* v_0 *and* c_0 *only, such that we may choose the Ricci flow* $g(t)$ *above to agree with the restriction of* $\tilde{g}(t)$ *to* $\mathbb{B}_{g_0}(x_0, k)$ *for times* $t \in [0, \min\{S, \tilde{S}_k, S_k\}]$*.*

Proof of Lemma 4.2.1. By making a uniform parabolic rescaling (scaling distances by a factor of 14), it suffices to prove the lemma under the apparently stronger hypothesis that $\tilde{g}(t)$ is assumed to be defined not just on $\mathbb{B}_{g_0}(x_0, k+1)$ but on the larger ball $\mathbb{B}_{g_0}(x_0, k+14)$, still satisfying the curvature decay $|\mathrm{Rm}|_{\tilde{g}(t)} \leq c_0/t$.

By Bishop-Gromov, for all $k \in \mathbb{N}$, there exists $v_k > 0$ depending only on k, α_0 and v_0 such that if $x \in \mathbb{B}_{g_0}(x_0, k+14)$ and $r \in (0, 1]$ then $\mathrm{Vol}\mathbb{B}_{g_0}(x, r) \geq v_k r^3$.

The first part of the lemma, giving the initial existence statement for $g(t)$, follows immediately by the local existence theorem 2.4.1 for some $C_k \geq 1$, $\alpha_k > 0$ and $S_k > 0$ depending only on α_0 and v_k, i.e. on α_0, k and v_0. We will allow ourselves to increase C_k and α_k, and decrease S_k, in order to establish the remaining claims of the lemma.

We increase each C_k to be at least as large as the constant C_0 retrieved from Lemma 2.4.5 with v_0 there equal to v_k here. Note that we are not actually applying Lemma 2.4.5, but simply retrieving a constant in preparation for its future application. By inductively replacing C_k by $\max\{C_k, C_{k-1}\}$ for $k = 2, 3, \ldots$, we can additionally assume that C_k is an increasing sequence. Thus C_k still depends only on k, α_0 and v_0, and in particular, not on c_0, and can be fixed for the remainder of the proof.

Suppose now that we would like to extend a Ricci flow $\tilde{g}(t)$. Appealing to the double bootstrap lemma 2.4.3 centred at each $x \in \mathbb{B}_{g_0}(x_0, k+12)$, there exists $\hat{S} > 0$ depending only on c_0 and α_0 so that for all $t \in [0, \min\{S, \hat{S}\}]$ we have $\mathrm{Ric}_{\tilde{g}(t)} \geq -100\alpha_0 c_0$ throughout $\mathbb{B}_{g_0}(x_0, k+12)$. (In due course, we will require a lower Ricci bound that does not depend on c_0.) In addition, after reducing $\hat{S} > 0$, still depending only on c_0 and α_0, the shrinking balls lemma 2.4.6 tells us that for all $x \in \mathbb{B}_{g_0}(x_0, k+10)$ we have $\mathbb{B}_{\tilde{g}(t)}(x, 1) \subset \mathbb{B}_{g_0}(x, 2) \subset \mathbb{B}_{g_0}(x_0, k+12)$ where the Ricci curvature is controlled, for all $t \in [0, \min\{S, \hat{S}\}]$.

Thus, for $x \in \mathbb{B}_{g_0}(x_0, k+10)$ we can apply Lemma 2.4.5 to deduce that $|\mathrm{Rm}|_{\tilde{g}(t)}(x) \leq C_k/t$ for all $t \in (0, \min\{S, \tilde{S}_k\}]$, for some $\tilde{S}_k \in (0, \hat{S}]$ depending only on v_k, α_0 and c_0, i.e only on k, c_0, v_0 and α_0.

Now we have a curvature decay estimate that does not depend on c_0 (albeit for a time depending on c_0) we can return to the double bootstrap lemma 2.4.3, which then tells us that on the smaller ball $\mathbb{B}_{g_0}(x_0, k+8)$ we have $\mathrm{Ric}_{\tilde{g}(t)} \geq -\alpha_k$ for $t \in [0, \min\{S, \tilde{S}_k\}]$, where α_k is increased to be at least $100\alpha_0 C_k$ and will be increased once more below (but only ever depending on k, α_0 and v_0) and where we have reduced $\tilde{S}_k > 0$ without adding any additional dependencies.

We can also exploit these new estimates to get better volume bounds via Lemma 2.4.10. We apply that result with $R = k + 8$ to obtain that for every $t \in [0, \min\{S, \tilde{S}_k\}]$, where we have reduced $\tilde{S}_k > 0$ again without adding any additional dependencies, we have $\mathbb{B}_{\tilde{g}(t)}(x_0, k+7) \subset \mathbb{B}_{g_0}(x_0, k+8)$, and for every $x \in \mathbb{B}_{\tilde{g}(t)}(x_0, k+6)$, we have $\mathrm{Vol}\mathbb{B}_{\tilde{g}(t)}(x, 1) \geq \varepsilon_k > 0$, where ε_k depends only on v_0, k, and α_0.

We need one final reduction of $\tilde{S}_k > 0$ in order to ensure appropriate nesting of balls defined at different times. By the expanding balls lemma 2.4.7, exploiting our lower Ricci bounds (even the weaker bound suffices here) we deduce that

$$\begin{cases} \mathbb{B}_{g_0}(x_0, k+4) \subset \mathbb{B}_{\tilde{g}(t)}(x_0, k+5) \\ \mathbb{B}_{g_0}(x_0, k+2) \subset \mathbb{B}_{\tilde{g}(t)}(x_0, k+3) \end{cases} \quad \text{for all } t \in [0, \min\{S, \tilde{S}_k\}], \tag{4.2.1}$$

with $\tilde{S}_k > 0$ reduced appropriately, without additional dependencies.

At this point we can fix $\tilde{S}_k$ and try to find our desired extension $g(t)$ of $\tilde{g}(t)$ by considering $\tilde{g}(\tau)$ for $\tau := \min\{S, \tilde{S}_k\}$ and restarting the flow from there. We cannot restart the flow using any variant of Shi's existence theorem (as was done in the extension lemma from [ST17], for example) since we would not have appropriate control on the existence time. Instead, we appeal to the local existence theorem 2.4.1. In order to do so, note that $\tilde{g}(\tau)$ satisfies the estimates $\mathrm{Ric}_{\tilde{g}(\tau)} \geq -\alpha_k$ on $\mathbb{B}_{\tilde{g}(\tau)}(x_0, k+7) \subset\subset \mathbb{B}_{g_0}(x_0, k+14)$, and $\mathrm{Vol}\mathbb{B}_{\tilde{g}(\tau)}(x, 1) \geq \varepsilon_k > 0$ for each $x \in \mathbb{B}_{\tilde{g}(\tau)}(x_0, k+6)$.

The output of the local existence theorem 2.4.1, applied with M there equal to $\mathbb{B}_{g_0}(x_0, k+14)$ here, with g_0 there equal to $\tilde{g}(\tau)$ here, and with $s_0 = k+7$, is that after reducing the $S_k > 0$ that we happened to find at the start of the proof, still depending only on α_0, k and v_0, there exists a Ricci flow $h(t)$ on $\mathbb{B}_{\tilde{g}(\tau)}(x_0, k+5)$ for $t \in [0, S_k]$, with $h(0) = \tilde{g}(\tau)$ where defined, and such that $\mathrm{Ric}_{h(t)} \geq -\alpha_k$ (after possibly increasing α_k, still depending only on α_0, k and v_0) and $|\mathrm{Rm}|_{h(t)} \leq c_k/t$, where c_k depends only on α_0, k and v_0. By the first inclusion of (4.2.1), this flow is defined throughout $\mathbb{B}_{g_0}(x_0, k+4)$.

Define a concatenated Ricci flow on $\mathbb{B}_{\tilde{g}(\tau)}(x_0, k+5) \supset \mathbb{B}_{g_0}(x_0, k+4)$ for $t \in [0, \tau + S_k]$ by

$$g(t) := \begin{cases} \tilde{g}(t) & 0 \leq t \leq \tau \\ h(t-\tau) & \tau < t \leq \tau + S_k. \end{cases} \tag{4.2.2}$$

This already satisfies the required lower Ricci bound $\mathrm{Ric}_{g(t)} \geq -\alpha_k$.

We claim that after possibly reducing S_k, without further dependencies, we have that for all $x \in \mathbb{B}_{g_0}(x_0, k+2)$, there holds the inclusion $\mathbb{B}_{g(t)}(x, 1) \subset\subset \mathbb{B}_{\tilde{g}(\tau)}(x_0, k+5)$ where the flow is defined, for all $t \in [0, \tau + S_k]$. But we already arranged that for $x \in \mathbb{B}_{g_0}(x_0, k+2) \subset \mathbb{B}_{g_0}(x_0, k+10)$ we have $\mathbb{B}_{\tilde{g}(t)}(x, 1) \subset \mathbb{B}_{g_0}(x, 2)$, which in turn is compactly contained in $\mathbb{B}_{g_0}(x_0, k+4) \subset \mathbb{B}_{\tilde{g}(\tau)}(x_0, k+5)$, so the claim holds up until time τ.

Thus to prove the claim it remains to show that for all $x \in \mathbb{B}_{g_0}(x_0, k+2)$, there holds the inclusion $\mathbb{B}_{h(t)}(x, 1) \subset\subset \mathbb{B}_{h(0)}(x_0, k+5)$ for all $t \in [0, S_k]$, and by the second inclusion of (4.2.1), it suffices to prove this for each $x \in \mathbb{B}_{h(0)}(x_0, k+3)$. But by the shrinking balls lemma 2.4.6, after reducing S_k depending on c_k, and thus on α_0, k and v_0, we can deduce that $\mathbb{B}_{h(t)}(x, 1) \subset \mathbb{B}_{h(0)}(x, 2) \subset\subset \mathbb{B}_{h(0)}(x_0, k+5)$ as required, thus proving the claim.

At this point we truncate the flow $g(t)$ to live only on the time interval $[0, S_k]$ (i.e. we chop off an interval of length τ from the end, not the beginning). The flow now lives on a time interval of length independent of c_0 and S.

The main final step is to apply Lemma 2.4.5 once more, with M there equal to $\mathbb{B}_{\tilde{g}(\tau)}(x_0, k+5)$ here. Using the claim we just proved, for every $x \in \mathbb{B}_{g_0}(x_0, k+2)$, after a possible further reduction of $S_k > 0$, and with C_k as fixed earlier, the local lemma 2.4.5 tells us that $|\mathrm{Rm}|_{g(t)}(x) \leq C_k/t$ for all $t \in (0, S_k]$. We finally have a sequence S_k that does what the lemma asks of it, except for being decreasing. The monotonicity of S_k and α_k can be arranged by iteratively replacing S_k by $\min\{S_k, S_{k-1}\}$, and α_k by $\max\{\alpha_k, \alpha_{k-1}\}$, for $k = 2, 3, \ldots$.

By restricting $g(t)$ to $\mathbb{B}_{g_0}(x_0, k)$ we are done. ∎

4.3. Pyramid Ricci Flow Construction - Proof of Theorem 4.1.2

Proof of Theorem 4.1.2. For our given constants α_0 and v_0, we appeal to Lemma 4.2.1 for increasing sequences $C_k \geq 1$ and $\alpha_k > 0$, and a decreasing sequence $S_k > 0$, all defined for $k \in \mathbb{N}$ and depending only on α_0 and v_0. Moreover, we construct a sequence $\tilde{S}_k$ as follows. For each $k \in \mathbb{N}$, we appeal to Lemma 4.2.1 with our given constants α_0 and v_0 and with $c_0 = C_{k+1}$. The sequences $C_k \geq 1$ and $\alpha_k > 0$ are suitable for the sequences required by the theorem.

An induction argument is required to get the constants T_k. We begin by setting T_1 to be S_1. The inductive step is as follows: Suppose we have picked $T_1, \ldots T_{k-1}$ for any integer $k \geq 2$. Then we set T_k to be the minimum of S_k, $\tilde{S}_{k-1}$ and T_{k-1}.

Note that when we pick T_k, it depends on S_k, i.e. on k, α_0 and v_0, and it also depends on $\tilde{S}_{k-1}$, i.e. additionally on C_k, but that itself only depends on k, α_0 and v_0.

Fix $l \in \mathbb{N}$. To construct $g_l(t)$, we appeal to Lemma 4.2.1 l times.

First we use the first part of that lemma with $k = l$. This initial flow lives on $\mathbb{B}_{g_0}(x_0, l)$ for a time S_l, and thus certainly for T_l.

Since $T_l \leq \tilde{S}_{l-1}$, we can appeal a second time to the lemma, this time with $k = l - 1$, in order to extend the flow $g_l(t)$ to the longer time interval $[0, T_{l-1}]$, albeit on the smaller ball $\mathbb{B}_{g_0}(x_0, l-1)$.

We repeat this process inductively for the remaining values of k down until it is finally repeated for $k = 1$. The resulting smooth Ricci flow $g_l(t)$ is now defined, for each $m \in \{1, \ldots, l\}$, on $\mathbb{B}_{g_0}(x_0, m)$ over the time interval $t \in [0, T_m]$, still satisfying that $g_l(0) = g_0$ where defined. Moreover, our repeated applications of Lemma 4.2.1 provide the estimates

$$\begin{cases} \mathrm{Ric}_{g_l(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_0}(x_0, m) \times [0, T_m] \\ |\mathrm{Rm}|_{g_l(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_0}(x_0, m) \times (0, T_m] \end{cases} \tag{4.3.1}$$

for each $m \in \{1, \ldots, l\}$, which completes the proof. ∎

4.4. Global-Local Mollification

Proof of Theorem 4.1.3. For the α_0 and v_0 of the theorem, we begin by retrieving sequences C_j, α_j and T_j from Theorem 4.1.2. Our first step is to modify them by throwing away the first two terms of each, i.e. replacing the three sequences by C_{j+2}, α_{j+2} and T_{j+2}.

With a view to later applying the shrinking balls lemma 2.4.6 for each $j \in \mathbb{N}$ we reduce T_j, without additional dependencies; with hindsight, it will suffice to ensure that

$$T_j < \frac{1}{4\beta^2 C_j} \tag{4.4.1}$$

where $\beta \geq 1$ is the universal constant arising in the shrinking balls lemma 2.4.6.

For each $i \in \mathbb{N}$ let $g_i(t)$ denote the *pyramid Ricci flow* obtained in Theorem 4.1.2 defined on a subset $\mathcal{D}_i \subset M \times [0,\infty)$ that now contains $\mathbb{B}_{g_0}(x_0, l+2) \times [0, T_l]$ for each $l \in \{1,\ldots,i\}$, having deleted the first two terms of the sequences C_j, α_j and T_j. If we fix $j \in \mathbb{N}$, then for $i \geq j$ the estimates of (4.1.1) hold for $g_i(t)$ on the g_0 ball of radius $j+2$. Consider an arbitrary point $x \in \mathbb{B}_{g_0}(x_0, j+1)$. We have the curvature estimate $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C_j}{t}$ throughout $\mathbb{B}_{g_0}(x,1) \times (0, T_j]$. The shrinking balls lemma 2.4.6 tells us that $\mathbb{B}_{g_i(t)}\left(x, \frac{1}{2}\right) \subset \mathbb{B}_{g_0}\left(x, \frac{1}{2} + \beta\sqrt{C_j T_j}\right) \subset\subset \mathbb{B}_{g_0}(x,1)$ for any $t \in [0, T_j]$, provided $\frac{1}{2} + \beta\sqrt{C_j T_j} < 1$. The restriction (4.4.1) ensures this is the case, and hence we establish that $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C_j}{t}$ throughout $\mathbb{B}_{g_i(t)}\left(x, \frac{1}{2}\right)$ for any $t \in (0, T_j]$.

These estimates allow us to repeat the argument of Miles Simon and Peter Topping in Theorem 1.7 in [ST17]. For completeness, we provide the details. Let

$$K_0 := 4 + \sup_{x \in \mathbb{B}_{g_0}(x_0, j+2)} \{|\mathrm{Rm}|_{g_0}(x)\} \in [4, \infty) \qquad \text{and} \qquad r_0 := \frac{1}{\sqrt{K_0}} \in \left(0, \frac{1}{2}\right]. \tag{4.4.2}$$

From (4.4.2) we may conclude that $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C_j}{t}$ throughout $\mathbb{B}_{g_i(t)}(x, r_0) \times (0, T_j]$, and that $|\mathrm{Rm}|_{g_0} \leq r_0^{-2}$ throughout $\mathbb{B}_{g_0}(x, r_0)$. Appealing to Lemma 2.5.5 allows us to deduce that $|\mathrm{Rm}|_{g_i(t)}(x) \leq B_1(\alpha_0, v_0, j, K_0)$ for all times $0 \leq t \leq T_j$. Repeating for every $x \in \mathbb{B}_{g_0}(x_0, j+1)$ allows us to conclude that $|\mathrm{Rm}|_{g_i(t)} \leq B_1$ throughout $\mathbb{B}_{g_0}(x_0, j+1) \times [0, T_j]$.

For each $q \in \mathbb{N}$ let $K_q := \sup\{|\nabla^q \mathrm{Rm}|_{g_0}(x) : x \in \mathbb{B}_{g_0}(x_0, j+2)\} \in (0,\infty)$ and define

$$r_q := \min\left\{B_1^{-\frac{1}{2}}, K_1^{-\frac{1}{3}}, \ldots, K_q^{-\frac{1}{2+q}}\right\} \in (0, r_0) \tag{4.4.3}$$

Let $z \in \mathbb{B}_{g_0}(x_0, j)$ and consider a fixed $q \in \mathbb{N}$. From (4.4.3) we may deduce that for every $l \in \{1,\ldots,q\}$ we have $|\nabla^l \mathrm{Rm}|_{g_0} \leq r_q^{-2-l}$ throughout $\mathbb{B}_{g_0}(z, r_q)$. Further, since $r_q \leq 1/2$ we have that $|\mathrm{Rm}|_{g_i(t)} \leq B_1 \leq r_q^{-2}$ throughout $\mathbb{B}_{g_0}(z, r_q) \times [0, T_j]$. A particular consequence of Lemma 2.5.6 is that there exists a constant $B_q \in (0,\infty)$, depending only on q, r_q and an upper bound for $T_j r_q^{-2}$, such that $|\nabla^q \mathrm{Rm}|_{g_i(t)}(z) \leq B_q$ for all times $0 \leq t \leq T_j$. Using (4.4.1) and (4.4.3), we have that $T_j r_q^{-2} \leq \frac{1}{4\beta^2 C_j} \max\left\{B_1, K_1^{\frac{2}{3}}, \ldots, K_q^{\frac{2}{2+q}}\right\}$, hence the constant B_q depends only on q, α_0, v_0, $K_0, \ldots, K_q$ and j, and in particular is independent of i. Repeating for every $z \in \mathbb{B}_{g_0}(x_0, j)$ allows us to conclude that $|\nabla^q \mathrm{Rm}|_{g_i(t)} \leq B_q$ throughout $\mathbb{B}_{g_0}(x_0, j) \times [0, T_j]$. Since $q \in \mathbb{N}$ was arbitrary, we can obtain such estimates for every $q \in \mathbb{N}$.

Armed with these curvature estimates we can work in coordinate charts and appeal to the Ascoli-Arzelá theorem to deduce that, after passing to a subsequence in i, we have smooth convergence $g_i(t) \to g(t)$, for some smooth Ricci flow $g(t)$ on $\mathbb{B}_{g_0}(x_0, j)$, defined for $t \in [0, T_j]$, with $g(0) = g_0$ on $\mathbb{B}_{g_0}(x_0, j)$, and satisfying the curvature estimates

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_j & \text{on } \mathbb{B}_{g_0}(x_0, j) \times [0, T_j] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_j}{t} & \text{on } \mathbb{B}_{g_0}(x_0, j) \times (0, T_j]. \end{cases} \tag{4.4.4}$$

We can now repeat this process for each $j = 1, 2, \ldots$ and take a diagonal subsequence to obtain a smooth limit Ricci flow $g(t)$ on a subset of spacetime that contains $\mathbb{B}_{g_0}(x_0, j) \times [0, T_j]$ for each $j \in \mathbb{N}$, with $g(0) = g_0$ throughout M, and satisfying (4.4.4) for every $j \in \mathbb{N}$. ■

4.5. Pyramid Ricci Flow Compactness Theorem

The following overarching theorem effectively includes Theorems 4.1.1, 4.1.4 and 4.1.5. The Ricci flows $g_k(t)$ arising here are pyramid Ricci flows coming from Theorem 4.1.2.

Theorem 4.5.1 (***Pyramid Ricci flow compactness; Theorem 5.1 in [MT18]***)**.** *Let $\left(\mathcal{M}_i^3, g_i, x_i\right)$ be a sequence of complete, smooth, pointed Riemannian three-manifolds such that for given $\alpha_0, v_0 > 0$ we have $\mathrm{Ric}_{g_i} \geq -\alpha_0$ throughout $\mathcal{M}_i$, and $\mathrm{Vol}\mathbb{B}_{g_i}(x_i, 1) \geq v_0 > 0$, for each $i \in \mathbb{N}$.*

Then there exist increasing sequences $C_k \geq 1$ and $\alpha_k > 0$ and a decreasing sequence $T_k > 0$, all defined for $k \in \mathbb{N}$, and depending only on α_0 and v_0, for which the following holds.

There exist a smooth three-manifold M, a point $x_0 \in M$, a complete distance metric $d : M \times M \to [0, \infty)$ generating the same topology as we already have on M, and a smooth Ricci flow $g(t)$ defined on a subset of spacetime $M \times (0, \infty)$ that contains $\mathbb{B}_d(x_0, k) \times (0, T_k]$ for each $k \in \mathbb{N}$, with $d_{g(t)} \to d$ locally uniformly on M as $t \downarrow 0$, and after passing to a subsequence in i we have that $(\mathcal{M}_i, d_{g_i}, x_i)$ converges in the pointed Gromov-Hausdorff sense to (M, d, x_0). Moreover, for any $k \in \mathbb{N}$,

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_k & \text{on } \mathbb{B}_d(x_0, k) \times (0, T_k] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_k}{t} & \text{on } \mathbb{B}_d(x_0, k) \times (0, T_k]. \end{cases} \tag{4.5.1}$$

Furthermore, for each $k \in \mathbb{N}$, there exist Ricci flows $g_k(t)$ defined on the subset of $\mathcal{M}_k \times [0, \infty)$ defined by

$$\mathcal{D}_k := \bigcup_{m=1}^{k} \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m],$$

with the properties that $g_k(0) = g_k$ on $\mathbb{B}_{g_k}(x_k, k+2)$ and

$$\begin{cases} \mathrm{Ric}_{g_k(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m] \\ |\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times (0, T_m], \end{cases} \tag{4.5.2}$$

for each $m \in \{1, \ldots, k\}$.

Moreover, for each $m \in \mathbb{N}$*, the flows* $g_k(t)$ *converge to* $(\mathbb{B}_d(x_0, m), g(t))$ *for* $t \in (0, T_m]$*, in the following sense: There exists a sequence of smooth maps* $f_k^m : \mathbb{B}_d(x_0, m) \to \mathbb{B}_{g_k}(x_k, m+1) \subset \mathcal{M}_k$*, mapping* x_0 *to* x_k*, such that for each* $\delta \in (0, T_m)$ *we have* $(f_k^m)^* g_k(t) \to g(t)$ *smoothly uniformly on* $B_d(x_0, m) \times [\delta, T_m]$.

Moreover, there exists a sequence of smooth maps $\varphi_k : \mathbb{B}_d(x_0, k) \to \mathbb{B}_{g_k}(x_k, k+1) \subset \mathcal{M}_k$*, diffeomorphic onto their images, mapping* x_0 *to* x_k*, such that, for any* $R > 0$*, as* $k \to \infty$ *we have the convergence*

$$d_{g_k}(\varphi_k(x), \varphi_k(y)) \to d(x, y)$$

uniformly as x, y *vary over* $\mathbb{B}_d(x_0, R)$*, and for sufficiently large* k*,* $\varphi_k|_{\mathbb{B}_d(x_0,R)}$ *is bi-Hölder with Hölder exponent depending only on* α_0*,* v_0 *and* R*. Moreover, for any* $r \in (0, R)$*, and for sufficiently large* k*,* $\varphi_k|_{\mathbb{B}_d(x_0,R)}$ *maps onto* $\mathbb{B}_{g_k}(x_k, r)$.

Finally, if g *is any smooth complete Riemannian metric on* M *then the identity map* $(M, d) \to (M, d_g)$ *is locally bi-Hölder.*

To clarify, by smooth uniform convergence, we mean uniform C^l convergence for arbitrary $l \in \mathbb{N}$. We remark that the bi-Hölder assertion for the maps φ_k in this theorem can be taken with respect to the distance metrics d and d_{g_k}, although one could replace g_k by any complete smooth metric.

Proof of Theorem 4.5.1. For the α_0 and v_0 of the theorem (as in Theorem 4.1.4) we begin by retrieving sequences C_j, α_j and T_j from Theorem 4.1.2.

Throughout the proof $\eta := \frac{1}{10}$ will be fixed. With a view to later applying Lemma 2.4.11 and both the expanding and shrinking balls lemmas, for each $j \in \mathbb{N}$ we reduce T_j, without additional dependencies, and with hindsight it will suffice to ensure that

$$\begin{cases} (i) \quad (4j+8)(1 - e^{-\alpha_j T_j}) < 1 - 8\beta\sqrt{C_j T_j} \qquad \text{(in particular } \beta\sqrt{C_j T_j} < \tfrac{1}{8}\text{) and} \\ (ii) \quad (j+1)(e^{\alpha_j T_j} - 1) \leq \eta \end{cases} \tag{4.5.3}$$

where $\beta \geq 1$ is the universal constant arising in the shrinking balls lemma 2.4.6. For $j \geq 2$, if necessary, we inductively replace T_j by $\min\left\{T_j, T_{j-1}, \frac{1}{j}\right\}$ to ensure the monotonicity of the sequence T_j remains, and to force $T_j \downarrow 0$ as $j \to \infty$.

We modify these sequences further by dropping the first two terms, i.e. by replacing each C_j, α_j and T_j by C_{j+2}, α_{j+2} and T_{j+2} respectively. This does not affect the monotonicity or dependencies. We may fix the values $C_j \geq 1$ and $\alpha_j > 0$ for each $j \in \mathbb{N}$ for the remainder of the proof. Before fixing T_j, we (potentially) reduce the value further.

With a view to appealing to Cheeger-Gromov-Hamilton compactness via Lemma 3.6.1, we reduce T_j, without additional dependencies, so that the conclusions of Lemma 3.6.1 for hypotheses $R = j+1$, $\eta = \frac{1}{10}$, $n = 3$, $v = v_0$, $\alpha = \alpha_j$ and $c_0 = C_j$ are valid for all times $t \in (0, T_j]$. As above, we may assume that T_j remains monotonically decreasing. After these reductions, we can now fix the value of T_j for each $j \in \mathbb{N}$ for the remainder of the proof.

For each $k \in \mathbb{N}$ let $g_k(t)$ denote the smooth *pyramid Ricci flow*, defined on the subset $\mathcal{D}_k \subset \mathcal{M}_k \times [0, \infty)$ obtained via Theorem 4.1.2. That is

$$\mathcal{D}_k = \bigcup_{m=1}^{k} \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m]. \tag{4.5.4}$$

(Recall that we have dropped the first two terms of the sequences, so we can work on a radius $m+2$ rather than m.) In particular, we have $g_k(0) = g_k$ where defined and for each $m \in \{1, \ldots, k\}$ we have

$$\begin{cases} \mathrm{Ric}_{g_k(t)} \geq -\alpha_m & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m] \\ |\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t} & \text{on } \mathbb{B}_{g_k}(x_k, m+2) \times (0, T_m]. \end{cases} \tag{4.5.5}$$

Fix $m \in \mathbb{N}$. For every $k \geq m$ the flow $g_k(t)$ is defined throughout $\mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m]$. Combining (4.5.5) with $\mathrm{Vol}\mathbb{B}_{g_k}(x_k, m+1) \geq v_0 > 0$ allows us to appeal to Lemma 3.6.1 with $R = m+1$, $\eta = \frac{1}{10}$, $n = 3$, $v = v_0$, $\alpha = \alpha_m$ and $c_0 = C_m$ to deduce that, after passing to a subsequence in k, we obtain a smooth three-manifold $\mathcal{N}_m$, a point $x_\infty^m \in \mathcal{N}_m$ and a smooth Ricci flow $\hat{g}_m(t)$ on $\mathcal{N}_m \times (0, T_m]$ with the following properties. First, for any $t \in (0, T_m]$ we have the inclusion

$$\mathbb{B}_{\hat{g}_m(t)}(x_\infty^m, m+1-\eta) \subset\subset \mathcal{N}_m. \tag{4.5.6}$$

Second, we have

$$\mathbb{B}_{\hat{g}_m(t)}(x_\infty^m, m+1-2\eta) \subset M_m, \tag{4.5.7}$$

for all $t \in (0, T_m]$, where M_m is the connected component of the interior of

$$\bigcap_{s \in (0, T_m]} \mathbb{B}_{\hat{g}_m(s)}(x_\infty^m, m+1-\eta) \subset \mathcal{N}_m \tag{4.5.8}$$

that contains x_∞^m. Combining (4.5.6) and (4.5.8) allows us to conclude that

$$M_m \subset\subset \mathcal{N}_m. \tag{4.5.9}$$

Moreover, Lemma 3.6.1 gives us a sequence of smooth maps $F_k^m : M_m \to \mathbb{B}_{g_k}(x_k, m+1) \subset \mathcal{M}_k$, for $k \geq m$, mapping x_∞^m to x_k, diffeomorphic onto their images and such that $(F_k^m)^* g_k(t) \to \hat{g}_m(t)$ smoothly uniformly on $M_m \times [\delta, T_m]$ as $k \to \infty$, for every $\delta \in (0, T_m)$.

Finally, we have

$$\begin{cases} \mathrm{Ric}_{\hat{g}_m(t)} \geq -\alpha_m & \text{on } M_m \times (0, T_m] \\ |\mathrm{Rm}|_{\hat{g}_m(t)} \leq \frac{C_m}{t} & \text{on } M_m \times (0, T_m] \,. \end{cases} \tag{4.5.10}$$

By taking an appropriate diagonal subsequence in k, we can be sure that these limits exist for *every* $m \in \mathbb{N}$.

We now wish to relate the limit flows $\hat{g}_m(t)$ that we have constructed, for different m. Let us fix m. Then $\hat{g}_m(T_{m+1})$ is a smooth limit of the metrics $g_k(T_{m+1})$ (modulo the diffeomorphisms F_k^m) defined on M_m. On the other hand, $\hat{g}_{m+1}(T_{m+1})$ is a smooth limit of the metrics $g_k(T_{m+1})$ (modulo the diffeomorphisms F_k^{m+1}) defined on M_{m+1}. Intuitively, M_{m+1} should be "bigger" than M_m since it arises from the compactness of the metrics on larger radius balls. This intuition is made precise in the following claim.

Claim: For sufficiently large k we have

$$F_k^m(M_m) \subset F_k^{m+1}(M_{m+1}). \tag{4.5.11}$$

Indeed, we have the stronger inclusion that for any $t \in (0, T_{m+1}]$ and sufficiently large k, depending on t,

$$F_k^m(M_m) \subset F_k^{m+1}\left(\mathbb{B}_{\hat{g}_{m+1}(t)}\left(x_\infty^{m+1}, m+2-2\eta\right)\right) \tag{4.5.12}$$

which immediately yields (4.5.11) via (4.5.7) by fixing $t = T_{m+1}$.

Proof: Recall that by definition of F_k^m, for all $k \geq m \in \mathbb{N}$ we have $F_k^m(M_m) \subset \mathbb{B}_{g_k}(x_k, m+1)$. For each $t \in (0, T_{m+1}]$, and sufficiently large k, depending on t, we may appeal to Part 2 of Lemma 3.3.3, with $2r = m+2-2\eta$, $b = 2r$, $a = m+2-3\eta$, $x_0 = x_\infty^{m+1}$, $(\mathcal{N}, \hat{g}) = (M_{m+1}, \hat{g}_{m+1}(t))$ and the sequence $\{\varphi_i\}$ being the sequence $\{F_k^{m+1}\}_{k \geq m+1}$, to deduce that $F_k^{m+1}(\mathbb{B}_{\hat{g}_{m+1}(t)}(x_\infty^{m+1}, m+2-2\eta)) \supset \mathbb{B}_{g_k(t)}(x_k, m+2-3\eta)$. Thus, in order to prove (4.5.12), it suffices to prove that

$$\mathbb{B}_{g_k}(x_k, m+1) \subset \mathbb{B}_{g_k(t)}(x_k, m+2-3\eta). \tag{4.5.13}$$

We prove this through a combination of the shrinking and expanding balls lemmas.

Recall from (4.5.5) we know that $\mathrm{Ric}_{g_k(t)} \geq -\alpha_m$ throughout $\mathbb{B}_{g_k}(x_k, m+2) \times [0, T_m]$ and $|\mathrm{Rm}|_{g_k(t)} \leq \frac{C_m}{t}$ throughout $\mathbb{B}_{g_k}(x_k, m+2) \times (0, T_m]$. Therefore we can appeal to the shrinking balls lemma 2.4.6 to deduce that $\mathbb{B}_{g_k(t)}(x_k, m+2-3\eta) \subset \mathbb{B}_{g_k}(x_k, m+2)$ provided $m+2-3\eta \leq m+2-\beta\sqrt{C_m t}$, which will be the case if $\beta\sqrt{C_m T_{m+1}} \leq 3\eta$, since $t \leq T_{m+1}$. But (i) in (4.5.3) tells us that $\beta\sqrt{C_m T_m} < \frac{1}{8}$, which is slightly stronger than required (recalling the monotonicity of the sequence T_j).

Thus we may conclude that $\mathrm{Ric}_{g_k(t)} \geq -\alpha_m$ throughout $\mathbb{B}_{g_k(t)}(x_k, m+2-3\eta) \times [0, T_{m+1}]$.

The expanding balls lemma 2.4.7 then tells us that $\mathbb{B}_{g_k(t)}(x_k, m+1+\eta) \supset \mathbb{B}_{g_k}(x_k, m+1)$, provided $(m+1+\eta)e^{-\alpha_m t} \geq m+1$, which will itself be true if $(m+1)(e^{\alpha_m T_{m+1}} - 1) \leq \eta$. Since $T_m \geq T_{m+1}$ we observe that (ii) of (4.5.3) ensures this is the case. But this inclusion is stronger than the inclusion (4.5.13) that we need. ††

By the uniqueness of smooth limits (i.e. Lemma 3.3.1) the metrics must agree in the sense that there is a smooth map $\psi_m : M_m \to M_{m+1}$ that is an isometry when domain and target are given the metrics $\hat{g}_m(T_{m+1})$ and $\hat{g}_{m+1}(T_{m+1})$ respectively, and which sends x_∞^m to x_∞^{m+1}.

Indeed, after passing to another subsequence, we could see ψ_m as a smooth limit, as $k \to \infty$, of maps $(F_k^{m+1})^{-1} \circ F_k^m$, which are well-defined because of the claim, and which are independent of time, and it is apparent that in fact ψ_m is an isometry also when domain and target are given the metrics $\hat{g}_m(t)$ and $\hat{g}_{m+1}(t)$ respectively, for any $t \in (0, T_{m+1}]$. Seeing ψ_m as such a limit and appealing to (4.5.12) allows us to conclude that

$$\psi_m(M_m) \subset \mathbb{B}_{\hat{g}_{m+1}(t)}\left(x_\infty^{m+1}, m+2-2\eta\right) \tag{4.5.14}$$

for any $t \in (0, T_{m+1}]$.

At this point we can already define a smooth extension of $\hat{g}_{m+1}(t)$ to the longer time interval $t \in (0, T_m]$, albeit on the smaller region $\psi_m(M_m)$, by taking $(\psi_m^{-1})^*(\hat{g}_m(t))$. However we would like to make such an extension for each m, and we must pause to construct the manifold on which this final flow will live.

The maps $\psi_m : M_m \to M_{m+1}$ allow us to apply Theorem 3.3.2 to the collection $\{M_m\}_{m\in\mathbb{N}}$. Doing so gives a smooth three-manifold M, a point $x_0 \in M$, and smooth maps $\theta_m : M_m \to M$, mapping x_∞^m to x_0, diffeomorphic onto their images, satisfying $\theta_m(M_m) \subset \theta_{m+1}(M_{m+1})$ and $\theta_{m+1}^{-1} \circ \theta_m = \psi_m$, and such that

$$M = \bigcup_{m\in\mathbb{N}} \theta_m(M_m). \tag{4.5.15}$$

In a moment, we will strengthen the inclusion $\theta_m(M_m) \subset \theta_{m+1}(M_{m+1})$ to assert that the images of M_m are contained within bounded subsets of M.

We can thus consider pull-back Ricci flows $(\theta_m^{-1})^*\hat{g}_m(t)$ on $\theta_m(M_m) \subset M$ for each m, and because ψ_m is an isometry, these pull-backs agree where they overlap. The union of the pull-backs we call $g(t)$. Moreover, the curvature estimates of (4.5.10) immediately give that for each $m \in \mathbb{N}$ we have

$$\begin{cases} \mathrm{Ric}_{g(t)} \geq -\alpha_m & \text{on } \theta_m(M_m) \times (0, T_m] \\ |\mathrm{Rm}|_{g(t)} \leq \frac{C_m}{t} & \text{on } \theta_m(M_m) \times (0, T_m]\,. \end{cases} \tag{4.5.16}$$

Furthermore, from (4.5.7) and (4.5.9) we have that

$$\mathbb{B}_{g(s)}(x_0, m+1-3\eta) = \theta_m(\mathbb{B}_{\hat{g}_m(s)}(x_\infty^m, m+1-3\eta)) \subset\subset \theta_m(M_m) \tag{4.5.17}$$

for any $0 < s \le T_m$.

Since $\theta_{m+1}^{-1} \circ \theta_m \equiv \psi_m$, (4.5.14) implies $\theta_{m+1}^{-1}(\theta_m(M_m)) \subset \mathbb{B}_{\hat{g}_{m+1}(t)}\left(x_\infty^{m+1}, m+2-2\eta\right) \subset M_{m+1}$ for any $t \in (0, T_{m+1}]$. Therefore we can strengthen the inclusion $\theta_m(M_m) \subset \theta_{m+1}(M_{m+1})$ to

$$\theta_m(M_m) \subset \mathbb{B}_{g(t)}(x_0, m+2-2\eta) \subset \theta_{m+1}(M_{m+1}) \tag{4.5.18}$$

for any $t \in (0, T_{m+1}]$.

For each $m \in \mathbb{N}$ we have a sequence $f_k^m : \theta_m(M_m) \to \mathbb{B}_{g_k}(x_k, m+1) \subset \mathcal{M}_k$ of smooth maps, for $k \ge m$, defined by $f_k^m := F_k^m \circ \theta_m^{-1}$, that map x_0 to x_k and are diffeomorphic onto their images. Moreover, from the choice of our diagonal subsequence, for any $\delta \in (0, T_m)$ we have

$$(f_k^m)^* \, g_k(t) \to g(t) \tag{4.5.19}$$

smoothly uniformly on $\theta_m(M_m) \times [\delta, T_m]$ as $k \to \infty$.

The obvious idea for constructing a distance metric d on M is to define $d := \lim_{t \downarrow 0} d_{g(t)}$, if we can be sure that this limit exists. The existence is a consequence of Lemma 2.4.11, which may be applied with $r = \frac{m}{2} + \frac{1}{4}$, $\alpha = \alpha_m$, $c_0 = C_m$ and $T = T_m$, which is possible due to the curvature estimates of (4.5.16), and the fact that from (4.5.17) we have $\mathbb{B}_{g(s)}(x_0, m+1-3\eta) \subset\subset \theta_m(M_m)$ for any $0 < s \le T_m$.

The result is a distance metric d on $\Sigma_m := \bigcap_{t \in (0, T_m]} \mathbb{B}_{g(t)}\left(x_0, \frac{m}{2} + \frac{1}{4}\right)$ arising as the uniform limit of $d_{g(t)}$ as $t \downarrow 0$. Moreover, for any $x, y \in \Sigma_m$ and any $0 < s \le T_m$ we have

$$d(x,y) - \beta\sqrt{C_m s} \le d_{g(s)}(x,y) \le e^{\alpha_m s} d(x,y) \tag{4.5.20}$$

and

$$\kappa_m(m, \alpha_0, v_0)\, [d(x,y)]^{1+4C_m} \le d_{g(s)}(x,y), \tag{4.5.21}$$

where $\kappa_m > 0$. As stated in Lemma 2.4.11, these estimates ensure d generates the same topology as we already have on Σ_m.

If we can estimate the R_0 from (2.4.17) by $R_0 > \frac{m}{2} + \frac{1}{8}$, then (2.4.17) gives that for any $t \in (0, T_m]$ we have

$$\mathbb{B}_d\left(x_0, \frac{m}{2} + \frac{1}{8}\right) \subset\subset \mathcal{O}_m \quad \text{and} \quad \mathbb{B}_{g(t)}\left(x_0, \frac{m}{2} + \frac{1}{8}\right) \subset\subset \Sigma_m \tag{4.5.22}$$

where $\mathcal{O}_m$ is the connected component of the interior of Σ_m that contains x_0. This lower bound for R_0 is true provided $\left(\frac{m}{2} + \frac{1}{4}\right) e^{-\alpha_m T_m} - \beta\sqrt{C_m T_m} > \frac{m}{2} + \frac{1}{8}$, i.e. if $1 - 8\beta\sqrt{C_m T_m} > (4m+2)(1 - e^{-\alpha_m T_m})$. Restriction (i) in (4.5.3) implies this inequality and hence the inclusions of (4.5.22) are valid.

A particular consequence of the first of these inclusions, via (4.5.20) and (4.5.21), is that for any $x, y \in \mathbb{B}_d(x_0, \frac{m}{2})$ and any $0 < s \leq T_m$ we have

$$d(x,y) - \beta\sqrt{C_m s} \leq d_{g(s)}(x,y) \leq e^{\alpha_m s} d(x,y) \tag{4.5.23}$$

and

$$\kappa_m(m, \alpha_0, v_0)\,[d(x,y)]^{1+4C_m} \leq d_{g(s)}(x,y). \tag{4.5.24}$$

The natural idea for extending d to the entirety of M is to repeat this procedure for all $m \in \mathbb{N}$. Of course this will require the sets $\{\Sigma_m\}_{m\in\mathbb{N}}$ to exhaust M. That this is indeed the case is a consequence of the following claim.

Claim: For every $m \in \mathbb{N}$ we have $\theta_m(M_m) \subset\subset \Sigma_{2m+4}$.

Proof: Recall from (4.5.18) we know that $\theta_m(M_m) \subset \mathbb{B}_{g(t)}(x_0, m+2-2\eta)$ for any $t \in (0, T_{m+1}]$. Moreover (4.5.22) gives that for any $t \in (0, T_{2m+4}]$ we have $\mathbb{B}_{g(t)}(x_0, m+2) \subset\subset \Sigma_{2m+4}$. Working with $t = T_{2m+4}$ in both of these inclusions gives the desired inclusion. ††

Knowing that the collection $\{\Sigma_m\}_{m\in\mathbb{N}}$ exhausts M allows us to repeat for all $m \in \mathbb{N}$ and extend d to the entirety of M whilst ensuring d generates the same topology as we already have on M.

Moreover, it is clear that (M, d) is a complete metric space. To elaborate, consider a Cauchy sequence in M with respect to d. This sequence is bounded and so contained within $\mathbb{B}_d\left(x_0, \frac{m}{2}\right)$ for some $m \in \mathbb{N}$. The first inclusion of (4.5.22) tells us that the closure of this ball is compact, so we may pass to a convergent subsequence. By virtue of the sequence being Cauchy, this establishes the sequence itself is convergent.

The estimates (4.5.23) and (4.5.24) give the local bi-Hölder regularity of the identity map on M that is claimed at the end of Theorem 4.5.1, as we now explain. Let $m \in \mathbb{N}$ and consider $\mathbb{B}_d\left(x_0, \frac{m}{2}\right) \subset\subset M$. For our arbitrary complete metric g on M, the distance metric d_g is bi-Lipschitz equivalent to $d_{g(T_m)}$ once restricted to $\mathbb{B}_d\left(x_0, \frac{m}{2}\right)$. The estimates (4.5.23) and (4.5.24) tell us that the identity map $\left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d\right) \to \left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d_{g(T_m)}\right)$ is Lipschitz continuous, whilst the identity map $\left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d_{g(T_m)}\right) \to \left(\mathbb{B}_d\left(x_0, \frac{m}{2}\right), d\right)$ is Hölder continuous, with Lipschitz constant and Hölder exponent depending only on α_0, v_0 and m. The arbitrariness of $m \in \mathbb{N}$ gives the desired local bi-Hölder regularity of the identity map $(M, d) \to (M, d_g)$.

Having d defined globally on M allows us to simplify several of the techniques utilised in [ST17]. For example, given $m \in \mathbb{N}$ the local uniform convergence of $d_{g(t)}$ to d as $t \downarrow 0$ tells us that for some $t_0 > 0$ we have $\mathbb{B}_d(x_0, m) \subset \mathbb{B}_{g(t)}\left(x_0, m + \frac{1}{2}\right)$ for every $t \in (0, \min\{t_0, T_m\}]$. Hence from (4.5.17) (recalling the definition of η)

$$\mathbb{B}_d(x_0, m) \subset\subset \theta_m(M_m) \tag{4.5.25}$$

and so the estimates of (4.5.16) are valid on $\mathbb{B}_d(x_0, m) \times (0, T_m]$. In fact, this establishes that the flow $g(t)$ lives where specified by the theorem.

We now turn our attention to defining the smooth maps φ_i. For each $m \in \mathbb{N}$, by (4.5.17) and (4.5.19) we have $(f_k^m)^* g_k(T_m) \to g(T_m)$ smoothly on $\overline{\mathbb{B}_{g(T_m)}(x_0, m+1-4\eta)}$ and so, by appealing to Lemma 3.3.3, we may choose $K(m)$ such that for all $k \geq K(m)$ we have

$$\left| d_{g_k(T_m)}\left(f_k^m(x), f_k^m(y)\right) - d_{g(T_m)}(x,y) \right| \leq \tfrac{1}{m}, \tag{4.5.26}$$

$$\left(1 + \tfrac{1}{m}\right)^{-1} d_{g(T_m)}(x,y) \leq d_{g_k(T_m)}\left(f_k^m(x), f_k^m(y)\right) \leq \left(1 + \tfrac{1}{m}\right) d_{g(T_m)}(x,y) \tag{4.5.27}$$

for all $x, y \in \mathbb{B}_{g(T_m)}\left(x_0, \frac{m}{2} + \frac{1}{4}\right)$, and

$$f_k^m\left(\mathbb{B}_{g(T_m)}\left(x_0, \tfrac{m}{2} - \tfrac{1}{2}\right)\right) \supset\supset \mathbb{B}_{g_k(T_m)}\left(x_k, \tfrac{m}{2} - \tfrac{3}{4}\right), \tag{4.5.28}$$

where (4.5.28) will be required later to ensure the image of the (not yet defined) map φ_i is large enough. We may assume that $K(m)$ is strictly increasing in m, otherwise we can fix $K(1)$, and then inductively replace $K(m)$ for $m = 2, 3, \ldots$ by the maximum of $K(m)$ and $K(m-1)+1$. Pass to a further subsequence in k by selecting the entries $K(1), K(2), K(3), \ldots$, so estimates (4.5.26), (4.5.27) and (4.5.28) now hold for all $k \geq m$.

For each $i \in \mathbb{N}$ we define a map $\varphi_i : \theta_i(M_i) \to \mathbb{B}_{g_i}(x_i, i+1) \subset \mathcal{M}_i$ by $\varphi_i := f_i^i$. In particular, each φ_i is defined throughout $\mathbb{B}_d(x_0, i)$ thanks to (4.5.25). These are diffeomorphisms onto their images, map x_0 to x_i and satisfy versions of the above estimates. Namely

$$\left| d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) - d_{g(T_i)}(x,y) \right| \leq \tfrac{1}{i}, \tag{4.5.29}$$

$$\left(1 + \tfrac{1}{i}\right)^{-1} d_{g(T_i)}(x,y) \leq d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) \leq \left(1 + \tfrac{1}{i}\right) d_{g(T_i)}(x,y) \tag{4.5.30}$$

for all $x, y \in \mathbb{B}_{g(T_i)}\left(x_0, \frac{i}{2} + \frac{1}{4}\right)$, and

$$\varphi_i\left(\mathbb{B}_{g(T_i)}\left(x_0, \tfrac{i}{2} - \tfrac{1}{2}\right)\right) \supset\supset \mathbb{B}_{g_i(T_i)}\left(x_i, \tfrac{i}{2} - \tfrac{3}{4}\right). \tag{4.5.31}$$

In what follows we will fix some $i_0 \in \mathbb{N}$ and consider the maps φ_i for $i \geq i_0$ restricted to the ball $\mathbb{B}_d(x_0, i_0)$. With this in mind we record the following observations.

Given a fixed $i_0 \in \mathbb{N}$, restriction (ii) in (4.5.3) (recalling the definition of η) ensures that $i_0 e^{\alpha_{i_0} T_{i_0}} < i_0 + \frac{1}{2}$. Hence (4.5.23) and the monotonicity of the sequence T_i imply that for all $i \geq i_0$ we have the inclusion

$$\mathbb{B}_d\left(x_0, \frac{i_0}{2}\right) \subset \mathbb{B}_{g(T_i)}\left(x_0, \frac{i_0}{2} + \frac{1}{4}\right). \tag{4.5.32}$$

This inclusion implies that for $i \geq i_0$ both (4.5.29) and (4.5.30) are valid for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$. Moreover, restriction (i) in (4.5.3) ensures that $\beta\sqrt{C_i T_i} < \frac{1}{8}$, and so (4.5.20) and (4.5.22) (with i here being used as the m there) yields that $\mathbb{B}_d\left(x_0, \frac{i}{2}\right) \supset \mathbb{B}_{g(T_i)}\left(x_0, \frac{i}{2} - \frac{1}{2}\right)$. Hence (4.5.31) implies $\varphi_i\left(\mathbb{B}_d\left(x_0, \frac{i}{2}\right)\right) \supset \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right)$.

Now we restrict φ_i to the ball $\mathbb{B}_d(x_0, i)$. Above we have shown that for any $i \in \mathbb{N}$ we have

$$\varphi_i\left(\mathbb{B}_d\left(x_0, \tfrac{i}{2}\right)\right) \supset\supset \mathbb{B}_{g_i(T_i)}\left(x_i, \tfrac{i}{2} - \tfrac{3}{4}\right). \tag{4.5.33}$$

Moreover, given $i_0 \in \mathbb{N}$ we have shown that for all $i \geq i_0$ we have

$$\left|d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) - d_{g(T_i)}(x, y)\right| \leq \tfrac{1}{i}, \tag{4.5.34}$$

$$\left(1 + \tfrac{1}{i}\right)^{-1} d_{g(T_i)}(x, y) \leq d_{g_i(T_i)}\left(\varphi_i(x), \varphi_i(y)\right) \leq \left(1 + \tfrac{1}{i}\right) d_{g(T_i)}(x, y) \tag{4.5.35}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$.

We now turn our attention to the properties of these maps restricted to balls of the form $\mathbb{B}_d(x_0, R)$. We first establish the uniform convergence and bi-Hölder regularity claims. For this purpose we take i_0 to be $i_0 := 2\left(\lfloor R \rfloor + 1\right) \in \mathbb{N}$.

For $i \geq i_0$ the *pyramid Ricci flow* $g_i(t)$ is defined on $\mathcal{D}_i$ (recall (4.5.4)), and in particular (4.5.5) gives that $\mathrm{Ric}_{g_i(t)} \geq -\alpha_{i_0}$ throughout $\mathbb{B}_{g_i}(x_i, i_0 + 2) \times [0, T_{i_0}]$ and $|\mathrm{Rm}|_{g_i(t)} \leq \frac{C_{i_0}}{t}$ throughout $\mathbb{B}_{g_i}(x_i, i_0 + 2) \times (0, T_{i_0}]$. But restriction (i) of (4.5.3) tells us that $\beta\sqrt{C_{i_0} T_{i_0}} < \frac{1}{8}$, so the shrinking balls lemma 2.4.6 gives that

$$\mathbb{B}_{g_i(s)}\left(x_i, i_0 + 2 - \tfrac{1}{8}\right) \subset \mathbb{B}_{g_i}\left(x_i, i_0 + 2 - \tfrac{1}{8} + \beta\sqrt{C_{i_0} T_{i_0}}\right) \subset\subset \mathbb{B}_{g_i}(x_i, i_0 + 2)$$

for any $s \in [0, T_{i_0}]$. These estimates allow us to apply Lemma 2.4.11 to the flow $g_i(t)$ with $r = \frac{i_0}{2} + 1 - \frac{1}{16}$, $n = 3$, $\alpha = \alpha_{i_0}$, $c_0 = C_{i_0}$ and $T = T_{i_0}$ to quantify the uniform convergence of $d_{g_i(s)}$ to d_{g_i} as $s \downarrow 0$ on $\Omega_i^{i_0} := \bigcap_{0 < t \leq T_{i_0}} \mathbb{B}_{g_i(t)}\left(x_i, \frac{i_0}{2} + 1 - \frac{1}{16}\right)$. Lemma 2.4.11 also gives that for any $z, w \in \Omega_i^{i_0}$ and any $0 < s \leq T_{i_0}$ we have

$$d_{g_i}(z, w) - \beta\sqrt{C_{i_0} s} \leq d_{g_i(s)}(z, w) \leq e^{\alpha_{i_0} s} d_{g_i}(z, w) \tag{4.5.36}$$

and

$$\gamma(i_0, \alpha_0, v_0)\left[d_{g_i}(z, w)\right]^{1 + 4C_{i_0}} \leq d_{g_i(s)}(z, w), \tag{4.5.37}$$

where $\gamma > 0$.

If R_0 from (2.4.17) satisfies $R_0 > \frac{i_0}{2} + \frac{1}{2}$, then (2.4.17) gives that

$$\mathbb{B}_{g_i(s)}\left(x_i, \frac{i_0}{2} + \frac{1}{2}\right) \subset\subset \Omega_i^{i_0} \tag{4.5.38}$$

for any $0 \leq s \leq T_{i_0}$, recalling that $g_i(0) = g_i$ on $\mathbb{B}_{g_i}(x_i, i+2)$. This lower bound for R_0 will be true provided $\left(\frac{i_0}{2} + 1 - \frac{1}{16}\right) e^{-\alpha_{i_0} T_{i_0}} - \beta\sqrt{C_{i_0} T_{i_0}} > \frac{i_0}{2} + \frac{1}{2}$. This inequality is itself true if $\frac{7}{2} - 8\beta\sqrt{C_{i_0} T_{i_0}} > \left(4i_0 + 8 - \frac{1}{2}\right)\left(1 - e^{-\alpha_{i_0} T_{i_0}}\right)$. Restriction (i) in (4.5.3) implies this latter inequality, and so the inclusions of (4.5.38) are valid.

We are now ready to establish the claimed uniform convergence. To do so we closely follow the argument of Miles Simon and Peter Topping utilised in the proof of Theorem 1.4 in [ST17].

Claim: As $i \to \infty$, we have convergence

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \to d(x,y) \qquad \text{uniformly as } x, y \text{ vary over } \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right). \tag{4.5.39}$$

Proof: Let $\varepsilon > 0$. We must make sure that for sufficiently large i, depending on ε, we have

$$|d_{g_i}(\varphi_i(x), \varphi_i(y)) - d(x,y)| < \varepsilon \tag{4.5.40}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$. By the distance estimates (4.5.36) and the inclusions of (4.5.38) there exists a $\tau_1 > 0$, depending only on $\varepsilon, i_0, \alpha_0$ and v_0, such that for all $i \geq i_0$ and any $s \in (0, \min\{\tau_1, T_{i_0}\}]$ we have

$$|d_{g_i}(z,w) - d_{g_i(s)}(z,w)| < \frac{\varepsilon}{3} \tag{4.5.41}$$

whenever there exists $t \in [0, T_{i_0}]$ such that $z, w \in \mathbb{B}_{g_i(t)}\left(x_i, \frac{i_0}{2} + \frac{1}{2}\right)$.

By the distance estimates (4.5.23) (for $m = i_0$) there exists a $\tau_2 > 0$, depending only on $\varepsilon, i_0, \alpha_0$ and v_0, such that for any $s \in (0, \min\{\tau_2, T_{i_0}\}]$ we have

$$|d(x,y) - d_{g(s)}(x,y)| < \frac{\varepsilon}{3} \tag{4.5.42}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$.

Let $\tau := \min\{\tau_1, \tau_2\} > 0$ (though we could have naturally picked the same τ_1 and τ_2 to begin with) and choose $i_1 \in \mathbb{N}$ such that for all $i \geq i_1$ we have $T_i < \tau$; this is possible since $T_i \downarrow 0$ as $i \to \infty$. Therefore for $i \geq \max\{i_0, i_1\}$ both (4.5.41) and (4.5.42) hold for $s = T_i$.

From (4.5.34), for all $i \geq \max\left\{i_0, \frac{3}{\varepsilon}\right\}$ we have

$$\left|d_{g_i(T_i)}(\varphi_i(x), \varphi_i(y)) - d_{g(T_i)}(x,y)\right| < \frac{1}{i} < \frac{\varepsilon}{3} \tag{4.5.43}$$

for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$.

Let $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and let $i \geq \max\left\{i_0, i_1, \frac{3}{\varepsilon}, 5\right\}$. Appealing to (4.5.32) gives $x, y \in \mathbb{B}_{g(T_i)}\left(x_0, \frac{i_0}{2} + \frac{1}{4}\right)$, thus (4.5.43) tells us that $\varphi_i(x), \varphi_i(y) \in \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i_0}{2} + \frac{1}{4} + \frac{1}{i}\right)$. Since $i \geq 5$ this tells us that (4.5.41) is valid for $z = \varphi_i(x)$ and $w = \varphi_i(y)$. Combining (4.5.41), (4.5.42) and (4.5.43) establishes (4.5.40) and completes the proof of the claim. ††

Since $\frac{i_0}{2} \geq R$, the uniform convergence on $\mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ gives uniform convergence on $\mathbb{B}_d(x_0, R)$.

The bi-Hölder estimates for $\varphi_i|_{\mathbb{B}_d(x_0,R)}$ are an easy consequence of those we have already obtained. If $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ then for $i \geq i_0$ (4.5.32) yields that $x, y \in \mathbb{B}_{g(T_i)}\left(x_0, \frac{i_0}{2} + \frac{1}{4}\right)$. Then (4.5.34) gives $\varphi_i(x), \varphi_i(y) \in \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i_0}{2} + \frac{1}{4} + \frac{1}{i}\right)$. Thus for $i \geq \max\{i_0, 5\}$ we have $\varphi_i(x), \varphi_i(y) \in \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i_0}{2} + \frac{1}{2}\right)$. Therefore by (4.5.38) both the estimates (4.5.36) and (4.5.37) are valid for $z = \varphi_i(x)$ and $w = \varphi_i(y)$.

As a first consequence we have, for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and all $i \geq \max\{i_0, 5\}$, that

$$\begin{aligned} d(x,y) &\overset{(4.5.24)}{\leq} \left[\frac{1}{\kappa_{i_0}(i_0, \alpha_0, v_0)} d_{g(T_i)}(x,y)\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(4.5.35)}{\leq} \left[\frac{\left(1+\frac{1}{i}\right)}{\kappa_{i_0}(i_0, \alpha_0, v_0)} d_{g_i(T_i)}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(4.5.36)}{\leq} \left[\frac{\left(1+\frac{1}{i}\right) e^{\alpha_{i_0} T_i}}{\kappa_{i_0}(i_0, \alpha_0, v_0)} d_{g_i}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}}. \end{aligned}$$

The monotonicity of the sequence T_i allows us to define $B(i_0, \alpha_0, v_0) := \left[\frac{2e^{\alpha_{i_0} T_{i_0}}}{\kappa_{i_0}(i_0,\alpha_0,v_0)}\right]^{\frac{1}{1+4C_{i_0}}} > 0$ and conclude that for all $i \geq \max\{i_0, 5\}$ we have

$$d(x,y) \leq B(i_0, \alpha_0, v_0) \left[d_{g_i}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}}. \tag{4.5.44}$$

Similarly, a second consequence is that for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and all $i \geq \max\{i_0, 5\}$ we have

$$\begin{aligned} d_{g_i}(\varphi_i(x), \varphi_i(y)) &\overset{(4.5.37)}{\leq} \left[\frac{1}{\gamma(i_0, \alpha_0, v_0)} d_{g_i(T_i)}(\varphi_i(x), \varphi_i(y))\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(4.5.35)}{\leq} \left[\frac{\left(1+\frac{1}{i}\right)}{\gamma(i_0, \alpha_0, v_0)} d_{g(T_i)}(x,y)\right]^{\frac{1}{1+4C_{i_0}}} \\ &\overset{(4.5.23)}{\leq} \left[\frac{\left(1+\frac{1}{i}\right) e^{\alpha_{i_0} T_i}}{\gamma(i_0, \alpha_0, v_0)} d(x,y)\right]^{\frac{1}{1+4C_{i_0}}}. \end{aligned}$$

The monotonicity of the sequence T_i allows us to define $A(i_0, \alpha_0, v_0) := \left[\frac{2e^{\alpha_{i_0} T_{i_0}}}{\gamma(i_0,\alpha_0,v_0)}\right] > 0$ and conclude that for all $i \geq \max\{i_0, 5\}$ we have

$$d_{g_i}(\varphi_i(x), \varphi_i(y)) \leq A(i_0, \alpha_0, v_0)^{\frac{1}{1+4C_{i_0}}} [d(x,y)]^{\frac{1}{1+4C_{i_0}}}. \tag{4.5.45}$$

Combining (4.5.44) and (4.5.45) yields that for all $x, y \in \mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ and all $i \geq \max\{i_0, 5\}$

$$\frac{[d_{g_i}(\varphi_i(x), \varphi_i(y))]^{1+4C_{i_0}}}{A(i_0, \alpha_0, v_0)} \leq d(x,y) \leq B(i_0, \alpha_0, v_0)\, [d_{g_i}(\varphi_i(x), \varphi_i(y))]^{\frac{1}{1+4C_{i_0}}}. \tag{4.5.46}$$

This establishes that for all $i \geq \max\{i_0, 5\}$ the restriction of φ_i to $\mathbb{B}_d\left(x_0, \frac{i_0}{2}\right)$ is bi-Hölder with Hölder exponent depending only on i_0, α_0 and v_0. Since $\frac{i_0}{2} \geq R$ and i_0 is determined by R, we deduce from (4.5.46) that, for all $i \geq \max\{i_0, 5\}$, the restriction of φ_i to $\mathbb{B}_d(x_0, R)$ is bi-Hölder with Hölder exponent depending only on α_0, v_0 and R as desired.

Next we turn our attention to the claim that the image of $\mathbb{B}_d(x_0, R)$ under φ_i is eventually arbitrarily close to being the whole of $\mathbb{B}_{g_i}(x_i, R)$. We know $\varphi_i\left(\mathbb{B}_d\left(x_0, \frac{i}{2}\right)\right) \supset \mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right)$ from (4.5.33). We claim that $\mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right) \supset \mathbb{B}_{g_i}\left(x_i, \frac{i}{2} - 1\right)$. To begin with we can appeal to the shrinking balls lemma 2.4.6 to deduce that

$$\mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right) \subset \mathbb{B}_{g_i}\left(x_i, \frac{i}{2} - \frac{3}{4} + \frac{1}{8}\right) \subset\subset \mathbb{B}_{g_i}(x_i, i+2)$$

since $1 - 8\beta\sqrt{C_i T_i} > 0$. This inclusion gives that $\mathrm{Ric}_{g_i(t)} \geq -\alpha_i$ throughout $\mathbb{B}_{g_i(T_i)}\left(x_i, \frac{i}{2} - \frac{3}{4}\right) \times [0, T_i]$. The expanding balls lemma 2.4.7 now gives our desired inclusion provided we have that $\left(\frac{i}{2} - \frac{3}{4}\right) e^{-\alpha_i T_i} \geq \frac{i}{2} - 1$, that is if $\left(i - \frac{3}{2}\right)\left(1 - e^{-\alpha_i T_i}\right) \leq \frac{1}{2}$. However this is guaranteed to be true by (ii) in (4.5.3), which imposed the stronger condition $(i+1)(e^{\alpha_i T_i} - 1) \leq \eta$. Therefore for all $i \geq 2(R+1)$ we have that

$$\varphi_i\left(\mathbb{B}_d(x_0, i)\right) \supset \varphi_i\left(\mathbb{B}_d(x_0, \frac{i}{2})\right) \supset \mathbb{B}_{g_i}(x_i, R). \tag{4.5.47}$$

Now suppose $r \in (0, R)$ as in the theorem. By the uniform convergence claim (4.5.39), we know that for sufficiently large i, let's say for $i \geq i_2$, we have $|d_{g_i}(\varphi_i(x), \varphi_i(y)) - d(x,y)| < \frac{R-r}{2}$ for all $x, y \in \overline{\mathbb{B}_d(x_0, R)}$, and in particular,

$$d(x_0, y) < d_{g_i}(x_i, \varphi_i(y)) + \frac{R-r}{2} \qquad \text{for all } y \in \overline{\mathbb{B}_d(x_0, R)}. \tag{4.5.48}$$

We claim that this implies our desired inclusion

$$\mathbb{B}_{g_i}(x_i, r) \subset \varphi_i(\mathbb{B}_d(x_0, R)) \qquad \text{for } i \geq i_2. \tag{4.5.49}$$

If not, then, keeping in mind (4.5.47), there exists $z \in \mathbb{B}_{g_i}(x_i, r)$ such that $y := \varphi_i^{-1}(z) \notin \mathbb{B}_d(x_0, R)$. Because we have $d(x_0, y) > R$, we can move a point $\hat{z}$ along a minimising geodesic from x_i to z until the first time that $d(x_0, \varphi_i^{-1}(\hat{z})) = R$, then replace z by $\hat{z}$. This guarantees that additionally we have $d(x_0, y) = R$ and $y \in \overline{\mathbb{B}_d(x_0, R)}$. But then by (4.5.48) we have

$$R = d(x_0, y) < d_{g_i}(x_i, z) + \frac{R - r}{2} < r + \frac{R - r}{2} < R, \tag{4.5.50}$$

a contradiction. Thus (4.5.49) holds as desired.

Finally we observe that, for sufficiently large $i \in \mathbb{N}$, slight modifications of the maps φ_i give ε-Gromov-Hausdorff approximations $\mathbb{B}_d(x_0, R)$ to $\mathbb{B}_{g_i}(x_i, R)$ (cf. Definition 2.8.2). Since $R > 0$ is arbitrary, we deduce that $(\mathcal{M}_i, d_{g_i}, x_i) \to (M, d, x_0)$ in the pointed Gromov-Hausdorff sense, defined in Definition 2.8.2, as $i \to \infty$. ∎

Chapter 5

Improved Pseudolocality on Large Hyperbolic Balls

This chapter is based upon [McL18], which is work completed by the author during his doctoral studies. Throughout this chapter, when referring to metric balls we use the convention that those denoted by $\mathbb{B}$ are taken to be open, whilst those denoted by $\overline{\mathbb{B}}$ are taken to be closed.

5.1. Introduction

An instructive simple setting for pseudolocality is when the initial metric is locally Euclidean on some ball. In particular, suppose we have a complete, smooth Ricci flow $g(t)$ on a smooth surface $\mathcal{M}^2$, defined for all $t \in [0, T]$ for some $T > 0$, with $\mathbb{B}_{g(0)}(x_0, R)$ isometric to a Euclidean disc of radius R. Then Theorem 2.6.3 gives a universal $A > 0$ such that for $0 \leq t \leq \min\{T, AR^2\}$ we have $\left|\mathrm{K}_{g(t)}(x_0)\right| \leq 2R^{-2}$. Therefore the Gauss curvature $\mathrm{K}_{g(t)}$ at the point x_0 remains close to 0 (the Euclidean Gauss curvature) for a time proportional to the square of the radius R.

In the hyperbolic setting, namely, when we have that $\mathbb{B}_{g(0)}(x_0, R)$ is isometric to a hyperbolic disc of radius R, Theorem 2.6.3 can again be applied. However, the requirement that $|\mathrm{K}_{g(0)}| \leq r_0^{-2}$ throughout $\mathbb{B}_{g(0)}(x_0, r_0)$ limits us to considering only radii $r_0 \in (0, 1]$. Therefore the Gauss curvature at x_0 may only be controlled for some fixed order one time, irrespective of how large R is.

Our first main result within this chapter establishes that, provided a sufficiently large initial ball is isometric to a hyperbolic disc of the same radius, the Gauss curvature at the central point remains bounded for a time that is exponential in the radius.

Theorem 5.1.1 (Improved control time with equality on large initial ball; Theorem 1.2 in [McL18]). *For any $\alpha \in (0,1]$ there exist constants $\mathcal{R} = \mathcal{R}(\alpha) > 0$ and $c = c(\alpha) > 0$ for which the following holds:*

Let $R \geq \mathcal{R}$ and assume that $g(t)$ is a complete smooth Ricci flow on a smooth surface $\mathcal{M}$, defined for all $t \in [0,T]$ for some $T > 0$, and such that, for some $x \in \mathcal{M}$, we have that $\left(\mathbb{B}_{g(0)}(x,R), g(0)\right)$ is isometric to a hyperbolic disc of radius R. Then at the point x we have

$$-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}}(x) \leq -1+\alpha \qquad \textit{for all} \qquad 0 \leq t \leq \mathcal{T}_{max} := \min\left\{T, e^{cR}\right\}. \tag{5.1.1}$$

Remark 5.1.2. Since the hyperbolic volume of a hyperbolic disc is exponential in the radius, by appealing to the well-developed two-dimensional existence theory (Theorem 2.3.2), we may deduce that $\left(\mathbb{B}_{g(0)}(x,R), g(0)\right)$ being isometric to a hyperbolic disc of radius R implies that the time T for which the flow exists may be taken to be exponential in the radius R. Therefore $\mathcal{T}_{max}$ in (5.1.1) can be assumed to be exponential in the radius R.

Remark 5.1.3. Given a complete hyperbolic surface $(\mathcal{M}, g_{\mathbb{H}})$, i.e. $\mathrm{K}_{g_{\mathbb{H}}} \equiv -1$ throughout $\mathcal{M}$ and $g_{\mathbb{H}}$ is complete, there is a unique complete Ricci flow $G(t) := (1+2t)g_{\mathbb{H}}$ with $G(0) \equiv g_{\mathbb{H}}$, and the Gauss curvature of this flow is $\mathrm{K}_{G(t)} \equiv -\frac{1}{1+2t}$. The uniqueness, a consequence of Theorem 1.1 in [Top15], allows us to refer to this flow as the hyperbolic Ricci flow on $\mathcal{M}$. Hence the Gauss curvature bound in (5.1.1) implies that the Gauss curvature at x remains C^0 close to the Gauss curvature of the hyperbolic Ricci flow for a time that is exponential in the radius R.

Remark 5.1.4. The completeness hypothesis can be weakened. The precise condition may be found in Theorem 5.5.1. Roughly, it requires $g(t)$ balls centred at points $z \in \mathbb{B}_{g(0)}(x,R)$ to remain compactly contained within $\mathcal{M}$, with the radius of the ball depending on the $g(0)$ distance of z from $\partial\mathbb{B}_{g(0)}(x,R)$. Of course a complete flow will automatically satisfy this condition.

Remark 5.1.5. We do not require the flow $g(t)$ to be of bounded curvature. This is a direct result of Theorem 2.6.3 being valid for flows with unbounded curvature. This is, to our knowledge, the only pseudolocality result valid for flows with unbounded curvature, and in dimensions $n \geq 3$ the unbounded curvature case of pseudolocality remains an interesting open question.

Since the pseudolocality result of Chen, Theorem 2.6.3, is applicable when the Gauss curvature of the initial metric $g(0)$ is only close to the Gauss curvature of the hyperbolic metric it is natural to wonder if our result remains valid under weakened almost-hyperbolic initial assumptions. The global situation suggests this should be the case. It is known that for Ricci flows conformally equivalent to complete hyperbolic metrics, if the initial metric is, in some sense, globally hyperbolic-like then the flow remains C^l close to the hyperbolic Ricci flow over its entire existence time. For example, see Theorem 2.3 in [GT11], and the subsequent discussion illustrating that the flows considered within this result may be extended to exist for all times $t \in [0, \infty)$.

Naturally, without assuming the desired Gauss curvature closeness at time $t = 0$, there must be some time delay before such an estimate becomes valid. Therefore we are led to expecting the result of Theorem 5.1.1 to be true, after an arbitrary short time delay, under weaker almost-hyperbolic assumptions at time $t = 0$. Our second main result verifies this expectation.

Theorem 5.1.6 (Improved control time under almost-hyperbolic hypotheses; Theorem 1.7 in [McL18]). *There is a universal $\varepsilon > 0$ such that for any $\alpha \in (0, 1]$ and any $\delta \in (0, \varepsilon)$ there exist constants $b = b(\alpha, \delta) \in (0, 1)$, $c = c(\alpha, \delta) > 0$ and $\mathcal{R} = \mathcal{R}(\alpha, \delta) > 0$ for which the following holds:*

Assume $R \geq \mathcal{R}$ and that $(\mathcal{M}, \mathcal{H})$ is a smooth surface with $\mathbb{B}_{\mathcal{H}}(x, R) \subset\subset \mathcal{M}$ for some $x \in \mathcal{M}$ and $(\mathbb{B}_{\mathcal{H}}(x, R), \mathcal{H})$ is isometric to a hyperbolic disc of radius R. Suppose $g(t)$ is a complete smooth Ricci flow on $\mathcal{M}$, defined for all $t \in [0, T]$ for some $T > 0$, with $g(0)$ conformal to $\mathcal{H}$ and satisfying that

$$(\mathbf{A}) \quad (1 - b)\mathcal{H} \leq g(0) \leq (1 + b)\mathcal{H} \qquad \textit{and} \qquad (\mathbf{B}) \quad |\mathrm{K}_{g(0)}| \leq 2 \tag{5.1.2}$$

throughout $\mathbb{B}_{\mathcal{H}}(x, R)$. Then at the point $x \in \mathcal{M}$ we have

$$-1 - \alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}}(x) \leq -1 + \alpha \qquad \textit{for all} \qquad \delta \leq t \leq \mathcal{T}_{max} := \min\left\{T, e^{cR}\right\}. \tag{5.1.3}$$

Remark 5.1.7. If $T < \delta$ then (5.1.3) is vacuous. However, the first estimate in (5.1.2) coupled with the fact that the hyperbolic volume of a hyperbolic disc is exponential in the radius yield that, for sufficiently large R, we have that $\mathrm{Vol}\mathbb{B}_{g(0)}(x, R) \geq e^{aR}$ for some universal $a > 0$. Therefore, as in Remark 5.1.2, it may be assumed that $\mathcal{T}_{max}$ in (5.1.3) is exponential in the radius R.

Remark 5.1.8. The Gauss curvature bound in (5.1.3) implies that, after an arbitrarily small delay, the Gauss curvature at x becomes C^0 close to the hyperbolic Gauss curvature, and remains so for a time that is exponential in the radius R.

Remark 5.1.9. The time $t = 0$ Gauss curvature bound of $|\mathrm{K}_{g(0)}| \leq 2$ throughout $\mathbb{B}_{\mathcal{H}}(0, R)$ could be weakened to being bounded by some $K_0 > 0$. However, the constant $\varepsilon > 0$ would now depend on K_0, and we must allow all the constants b, c and $\mathcal{R}$ to additionally depend on K_0.

Remark 5.1.10. As in Remark 5.1.5 we do not require the flow $g(t)$ to be of bounded curvature. Moreover, completeness of the flow $g(t)$ can be weakened as alluded to in Remark 5.1.4.

The techniques used to prove our main results exploit many advantageous facts about Ricci flow specific to dimension 2 (cf. Section 5.2). Hence they cannot generalise to higher dimensions. However, there are no obvious non-artificial obstructions to the higher dimensional analogues, and we make the following conjecture that the same phenomenon is valid in higher dimensions.

Conjecture 2 (**Improved time control with equality on initial ball; Conjecture 1 in [McL18]**). *Let $n \in \mathbb{N}$ such that $n \geq 3$. There are constants $\mathcal{A} = \mathcal{A}(n) > 0$, $c = c(n) > 0$ and $\mathcal{R} = \mathcal{R}(n) > 0$ for which the following holds:*

Let $R \geq \mathcal{R}$ and suppose that $g(t)$ is a smooth complete Ricci flow of bounded curvature on a smooth n-dimensional manifold $\mathcal{M}$, defined for all $t \in [0,T]$ for some $T > 0$, and, for some $x \in \mathcal{M}$, suppose we have that $\left(\mathbb{B}_{g(0)}(x,R), g(0)\right)$ is isometric to a hyperbolic ball of radius R. Then at $x \in \mathcal{M}$ we have that

$$|\mathrm{Rm}|_{g(t)}(x) \leq \mathcal{A} \qquad \textit{for all} \qquad 0 \leq t \leq \mathcal{T}_{max} := \min\{T, e^{cR}\}.$$

We further expect that the hypotheses of the previous conjecture can be weakened to almost-hyperbolic hypotheses in a similar spirit to the hypotheses of Theorem 5.1.6. The remainder of this Chapter is structured as follows. In Section 5.2 we collect together several well-known facts about two-dimensional Ricci flow and hyperbolic geometry. In Section 5.3 we state some PDE regularity results, which can all be found in [LSU68] for example, that we will require in subsequent sections. In Section 5.4 we prove several supplementary lemmata recording how (and in what sense) our local almost-hyperbolic hypotheses are preserved under Ricci flow. Finally in Section 5.5 we provide proof of both Theorem 5.1.1 and Theorem 5.1.6. In fact, both results are simple consequences of Theorem 5.5.1.

5.2. Ricci Flow on Surfaces

On a smooth two-dimensional surface we have that $\mathrm{Ric}_g = \mathrm{K}_g \cdot g$. Thus the Ricci flow equation (2.3.1) becomes

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{K}_{g(t)} \cdot g(t). \tag{5.2.1}$$

Therefore the Ricci flow moves within a fixed conformal class. If we pick a local isothermal complex coordinate $z = x + iy$ on $U \subset \mathcal{M}$ we can write the metric (on U) as $g = e^{2u}|dz|^2$ for a scalar conformal factor $u \in C^\infty(U)$. A simple computation shows that the evolution of the metric's conformal factor on U under Ricci flow satisfies

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u = -\mathrm{K}_{g(t)} \tag{5.2.2}$$

where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is defined with respect to the local coordinate $z = x + iy$.

Let h be the complete conformal metric of constant Gauss curvature -1 on $\mathbb{D} := \{z \in \mathbb{C} :$

$|z| < 1\}$ which may be globally written as $h = e^{2\varphi}|dz|^2$ where $\varphi(z) := \log \frac{2}{1-|z|^2}$. Throughout we work on smooth surfaces $(\mathcal{M}, \mathcal{H})$ that contain a point $x \in \mathcal{M}$ such that for some $R > 0$ the ball $\mathbb{B}_{\mathcal{H}}(x, R) \subset\subset \mathcal{M}$ and we have that $(\mathbb{B}_{\mathcal{H}}(x, R), \mathcal{H})$ is isometric to a hyperbolic disc of radius R, i.e. to $(\mathbb{B}_h(0, R), h)$. Clearly any smooth Ricci flow $g(t)$ defined on $\mathbb{B}_{\mathcal{H}}(x, R)$ for all $t \in [0, T]$ may be viewed as a smooth Ricci flow defined on $\mathbb{B}_h(0, R)$ for all $t \in [0, T]$.

Suppose that, for some $w \in \mathbb{D}$ and $r > 0$, we have a smooth Ricci flow $g(t)$ defined on $\mathbb{B}_h(w, r)$ for all $t \in [0, T]$. By choosing a local isothermal complex coordinate z, we can write $g = e^{2u}|dz|^2$ throughout $\mathbb{B}_h(w, r) \times [0, T]$ for a smooth scalar function $u : \mathbb{B}_h(w, r) \times [0, T] \to \mathbb{R}$. Choosing a different local isothermal complex coordinate will induce a different conformal factor, however, the difference of two conformal factors is invariantly defined.

Given any $w \in \mathbb{D}$ we may choose a Möbius diffeomorphism (an isometry of $\mathbb{D}$ with respect to the hyperbolic metric h) mapping 0 to w. We will frequently exploit this and pull back via such a diffeomorphism to reduce working near a point $w \in \mathbb{D}$ to working near the origin $0 \in \mathbb{D}$. In view of the invariance of the difference of two conformal factors, and since h is invariantly defined, we see that any estimates on the difference of two Ricci flow's conformal factors with respect to the metric h is preserved under such pull backs.

One particularly important example of this for our purposes is the pointwise difference between the Gauss curvature K_g of a metric g conformally equivalent to h and the Gauss curvature K_h of the hyperbolic metric h itself. Further, if we let u be a conformal factor for g, so that $g = e^{2u}|dz|^2$, then we can compute that

$$\mathrm{K}_g - \mathrm{K}_h = -e^{-2u}\Delta u + e^{-2\varphi}\Delta\varphi = -e^{-2(u-\varphi)}\Delta_h(u - \varphi) + (1 - e^{-2(u-\varphi)}), \tag{5.2.3}$$

where we have recalled that $-1 \equiv \mathrm{K}_h = -e^{-2\varphi}\Delta\varphi = -\Delta_h\varphi$. The particular form given in (5.2.3) will be useful later in Section 5.4

Frequently it will be convenient to switch between the hyperbolic distance from 0 and the Euclidean distance from 0 on $\mathbb{D}$. For any $z \in \mathbb{D}$ we have $d_h(0, z) = \log\left[\frac{1+|z|}{1-|z|}\right] = 2\tanh^{-1}(|z|)$ and hence $\mathbb{B}_h(0, R) = \mathbb{D}_{\tanh(R/2)}$. Here we use the notation that $\mathbb{D}_\rho := \{z \in \mathbb{D} : |z| < \rho\}$ for $0 < \rho < 1$. With a view to later requiring lower bounds on certain radii, we record the following elementary lower bound for $\tanh$.

Lemma 5.2.1 (Elementary lower bound for $\tanh$; Lemma 2.1 in [McL18])**.** *For any $x \in (0, \infty)$ we have the lower bound*

$$\tanh(x) \geq 1 - \frac{1}{x}. \tag{5.2.4}$$

Proof of Lemma 5.2.1. Define $F : (0, \infty) \to (0, 1)$ by $F(x) := x\tanh(x) - x + 1$. It suffices to establish that $F(x) \geq 0$ throughout $(0, \infty)$. Since $\tanh(x) > 0$ on $(0, \infty)$ it is apparent that

$F(x) > 0$ for every $x \in (0,1)$. For $x \geq 1$ we compute the derivative of F and observe

$$F'(x) = \tanh(x) - 1 + x\,\mathrm{sech}^2(x) = \frac{(4x-2)e^{2x} - 2}{(e^{2x}+1)^2} \geq 0.$$

Thus, for $x \geq 1$, we have that $F(x) \geq F(1) = \tanh(1) > 0$. Therefore $F(x) > 0$ for all $x \in (0,\infty)$. ∎

Whilst working on the disc $\mathbb{D}$, we will occasionally need to convert between C^l bounds with respect to the hyperbolic metric h and the Euclidean metric g_E. To do so we will use the following well known result.

Lemma 5.2.2 (Equivalent C^k norms; Lemma B.5 in [GT11])**.** *Let h denote the complete conformal metric of constant Gauss curvature -1 on $\mathbb{D}$ and P be an arbitrary smooth (r,q) tensor field on $\mathbb{D}$. Then given any $l \in \mathbb{N}_0$ and $\rho \in (0,1)$ there exists a constant $C = C(l,\rho,r,q) > 0$ for which*

$$\frac{1}{C}||P||_{C^l(\mathbb{D}_\rho;g_E)} \leq ||P||_{C^l(\mathbb{D}_\rho;h)} \leq C||P||_{C^l(\mathbb{D}_\rho;g_E)} \tag{5.2.5}$$

where g_E is the flat Euclidean metric on $\mathbb{D}$. In particular, at $0 \in \mathbb{D}$ we have

$$\frac{1}{C}\sum_{k=0}^{l}|\nabla^k_{g_E}P|_{g_E}(0) \leq \sum_{k=0}^{l}|\nabla^k_h P|_h(0) \leq C\sum_{k=0}^{l}|\nabla^k_{g_E}P|_{g_E}(0). \tag{5.2.6}$$

Finally, recall the following elementary weak comparison principle, found in [Gie12], for example.

Theorem 5.2.3 (Elementary comparison principle; Theorem 2.3.1 in [Gie12] and Theorem 2.2 in [McL18])**.** *Let $\mathcal{U} \subset \mathbb{C}$ be an open, bounded domain and, for some $T > 0$, suppose $w, v \in C^\infty(\overline{\mathcal{U}} \times [0,T])$ both be solutions of the equation $\frac{\partial\psi}{\partial t} = e^{-2\psi}\Delta\psi$ throughout $U \times [0,T]$. If $v(z,0) \geq w(z,0)$ throughout $\mathcal{U}$ and $v(z,t) \geq w(z,t)$ throughout $\partial\mathcal{U} \times [0,T]$ then we may conclude that $v(z,t) \geq w(z,t)$ throughout $\overline{\mathcal{U}} \times [0,T]$.*

5.3. Regularity Theory

Given some interval $\mathcal{T} \subset [0,\infty)$, including 0, and some domain $\mathcal{M} \subset\subset \mathbb{R}^n$ there is the following notion of *parabolic distance* $\mathrm{dist}_p((z,t),(w,s)) := |z-w| + \sqrt{|t-s|}$ on $\mathcal{M} \times \mathcal{T}$ and of the *parabolic boundary* $\partial_p(\mathcal{M} \times \mathcal{T}) := (\partial\mathcal{M} \times \mathcal{T}) \cup (\mathcal{M} \times \{0\})$ of $\mathcal{M} \times \mathcal{T}$.

Fix $\alpha \in (0,1)$. The *parabolic Hölder space* $C^{\alpha,\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T})$ is the space of functions $f \in C^0\left(\overline{\mathcal{M} \times \mathcal{T}}\right)$ for which

$$[f]_{C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})} := \sup_{(z,t)\neq(w,s)\in\mathcal{M}\times\mathcal{T}} \left\{ \frac{|f(z,t)-f(w,s)|}{\mathrm{dist}_p\left((z,t),(w,s)\right)^\alpha} \right\} < \infty. \tag{5.3.1}$$

The $C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})$-norm is then given by

$$||f||_{C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})} := [f]_{C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})} + ||f||_{C^0(\mathcal{M}\times\mathcal{T})}. \tag{5.3.2}$$

Under this norm $C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})$ is a Banach space (i.e. a complete normed vector space).

Given $k\in\mathbb{N}$ the spaces $C^{2k+\alpha,k+\frac{\alpha}{2}}(\mathcal{M},\times\mathcal{T})$ are defined similarly. If $\gamma=(\gamma_1,\ldots,\gamma_n)\in\mathbb{N}^n$ is a multi-index then $|\gamma|:=\gamma_1+\ldots+\gamma_n$ and we set $\partial^\gamma := \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\partial x_2^{\gamma_2}\ldots\partial x_n^{\gamma_n}}$. Then

$$f\in C^{2k+\alpha,k+\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T}) \qquad \Longleftrightarrow \qquad \frac{\partial^r}{\partial t^r}\partial^\gamma f\in C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T}) \tag{5.3.3}$$

for all $r\in\mathbb{N}$ and $\gamma\in\mathbb{N}^n$ for which $2r+|\gamma|\leq 2k$. The $C^{2k+\alpha,k+\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})$-norm is defined by

$$||f||_{C^{2k+\alpha,k+\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})} := \sum_{2r+|\gamma|\leq 2k}\left|\left|\frac{\partial^r}{\partial t^r}\partial^\gamma f\right|\right|_{C^{\alpha,\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})} \tag{5.3.4}$$

which makes $C^{2k+\alpha,k+\frac{\alpha}{2}}(\mathcal{M}\times\mathcal{T})$ into a Banach space.

With the notation all introduced we can turn our attention to the PDE regularity theory we will utilise. The following result establishes that the Hölder norm of a bounded solution to a quasi-linear parabolic PDE away from the parabolic boundary is controlled by the L^∞ norm over the entire region of spacetime. The result itself is a simplified variant of Theorem V.1.1 in [LSU68], though our notation and formulation is more in line with Theorem B.1.1 in [Gie12].

Theorem 5.3.1 (Simplified variant of Theorem V.1.1 in [LSU68]; also see Theorem B.1.1 in [Gie12])**.** *Let $\mathcal{T}$ be an interval such that $0\in\mathcal{T}\subset[0,\infty)$ and $\mathcal{M}\subset\subset\mathbb{R}^n$ be a domain. Let $\lambda^{-1},\Lambda,\beta:[0,\infty)\to(0,\infty)$ all be continuous monotonically increasing functions. Suppose $u\in C^{2,1}(\mathcal{M}\times\mathcal{T})$ is a solution of*

$$\frac{\partial u}{\partial t}(z,t) = \operatorname{div}\mathbf{A}\left[z,t,u(z,t),Du(z,t)\right] + B(z,t,u(z,t),Du(z,t)) \tag{5.3.5}$$

where $\mathbf{A}\in C^0(\mathcal{M}\times\mathcal{T}\times\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$ and $B\in C^0(\mathcal{M}\times\mathcal{T}\times\mathbb{R}\times\mathbb{R}^n)$. Further assume that for all $(z,t,w,\xi)\in\mathcal{M}\times\mathcal{T}\times\mathbb{R}\times\mathbb{R}^n$ we have

- $\lambda(|w|)|\xi|^2\leq\langle\mathbf{A}(z,t,w,\xi),\xi\rangle$,
- $|\mathbf{A}(z,t,w,\xi)|\leq\Lambda(|w|)|\xi|$ *and*
- $|B(z,t,w,\xi)|\leq\beta\left(|w|\right)|\xi|^2$.

Finally assume that $M:=||u||_{L^\infty(\mathcal{M}\times\mathcal{T})}<\infty$.

Then given any $\delta>0$ there exist constants $\alpha=\alpha\left(n,\lambda(M),\Lambda(M),\frac{M\beta(M)}{\lambda(M)}\right)\in(0,1]$ and $C_0=C_0(n,M,\lambda(M),\Lambda(M),\beta(M),\delta)\in(0,\infty)$ such that for any $\mathcal{K}\subset\subset\mathcal{M}\times\mathcal{T}$ with

$\mathrm{dist}_p(\mathcal{K}, \partial_p(\mathcal{M} \times \mathcal{T})) \geq \delta$ *we have*

$$||u||_{C^{\alpha, \frac{\alpha}{2}}(\mathcal{K})} \leq C_0. \tag{5.3.6}$$

If for some $\kappa \in (0,1)$ *we have that* $u(\cdot, 0) \in C^{\kappa}(\mathcal{M})$ *then we need only avoid the spatial part of the parabolic boundary. To be more precise, given any* $\mathcal{U} \subset\subset \mathcal{M}$ *with* $\delta_1 := \mathrm{dist}(\mathcal{U}, \partial\mathcal{M}) > 0$ *there is a constant* $\alpha_1 = \alpha_1\left(n, \kappa, \delta_1, \lambda(M), \Lambda(M), \frac{M\beta(M)}{\lambda(M)}\right) \in (0,1]$ *for which*

$$||u||_{C^{\alpha_1, \frac{\alpha_1}{2}}(\mathcal{U} \times \mathcal{T})} \leq C_1 = C_1\left(n, M, \lambda(M), \Lambda(M), \beta(M), \delta_1, \kappa, ||u(\cdot, 0)||_{C^{\kappa}(\mathcal{M})}\right) < \infty.$$

The following result allows us to bootstrap and improve the Hölder regularity of solutions to linear parabolic PDE away from the parabolic boundary. It is a special case of Theorem IV.10.1 in [LSU68]. Again our notation and formulation is more in line with Theorem B.2.1 in [Gie12].

Theorem 5.3.2 (Parabolic Schauder estimates; Theorem B.2.1 in [Gie12])**.** *Let* $\mathcal{T} \subset [0, \infty)$ *be an interval,* $\mathcal{M} \subset\subset \mathbb{R}^n$ *a domain,* $r, q \in \mathbb{N}_0$ *and* $\alpha \in (0,1]$ *all be given. Assume* $u \in C^{2r+2+\alpha, q+1+\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T})$ *is a solution of the equation*

$$\frac{\partial u}{\partial t}(z,t) = \langle \mathbf{a}(z,t), \mathrm{Hess}(u)(z,t) \rangle + \psi(z,t) \tag{5.3.7}$$

for given $\mathbf{a} \in C^{2r+\alpha, q+\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T}; \mathrm{Sym}_2 \mathbb{R}^n)$ *and* $\psi \in C^{2r+\alpha, q+\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T})$. *Further suppose that for any* $(z,t) \in \mathcal{M} \times \mathcal{T}$ *and any* $\xi \in \mathbb{R}^n \setminus \{0\}$ *we have that* $\langle \mathbf{a}(z,t), \xi \otimes \xi \rangle \geq \rho |\xi|^2$ *for some* $\rho > 0$.

Given any $\mathcal{K} \subset\subset \mathcal{M} \times \mathcal{T}$ *with* $\delta := \mathrm{dist}_p(\mathcal{K}, \partial_p(\mathcal{M} \times \mathcal{T})) > 0$ *there exists*

$$C_0 = C_0\left(n, \rho, r, q, \alpha, \delta, ||\mathbf{a}||_{C^{2r+\alpha, q+\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T}; \mathrm{Sym}_2 \mathbb{R}^n)}, ||\psi||_{C^{2r+\alpha, q+\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T})}\right) \in (0, \infty)$$

such that

$$||u||_{C^{(2r+2)+\alpha, (q+1)+\frac{\alpha}{2}}(\mathcal{K})} \leq C_0 ||u||_{L^{\infty}(\mathcal{M} \times \mathcal{T})}. \tag{5.3.8}$$

If $0 \in \mathcal{T}$ *and* $||u(\cdot, 0)||_{C^{(2r+2)+\alpha}(\mathcal{M})} < \infty$ *then we need only avoid the spatial part of the parabolic boundary. To be more precise, given any* $\mathcal{U} \subset\subset \mathcal{M}$ *with* $\delta_1 := \mathrm{dist}(\mathcal{U}, \partial\mathcal{M}) > 0$ *there exists*

$$C_1 = C_1\left(n, \rho, r, q, \alpha, \delta_1, ||\mathbf{a}||_{C^{2r+\alpha, q+\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T}; \mathrm{Sym}_2 \mathbb{R}^n)}, ||\psi||_{C^{2r+\alpha, q+\frac{\alpha}{2}}(\mathcal{M} \times \mathcal{T})}\right) \in (0, \infty)$$

such that

$$||u||_{C^{(2r+2)+\alpha,(q+1)+\frac{\alpha}{2}}(\mathcal{U}\times\mathcal{T})} \leq C_1 \left(||u(\cdot,0)||_{C^{(2r+2)+\alpha}(\mathcal{M})} + ||u||_{L^\infty(\mathcal{M}\times\mathcal{T})}\right). \tag{5.3.9}$$

If $v \in C^{p,q}(\mathcal{M}\times\mathcal{T})$ is a solution to the quasi-linear PDE

$$\frac{\partial v}{\partial t}(z,t) = \left\langle \tilde{\mathbf{a}}(z,t,v(z,t),Dv(z,t)), \mathrm{Hess}(v)(z,t)\right\rangle + \tilde{\psi}(z,t,v(z,t),Dv(z,t)) \tag{5.3.10}$$

then, defining $\mathbf{a} \in C^{p-1,q}\left(\mathcal{M}\times\mathcal{T}; \mathrm{Sym}_2\,\mathbb{R}^n\right)$ by $\mathbf{a}(z,t) := \tilde{\mathbf{a}}(z,t,v(z,t),Dv(z,t))$ and $\psi \in C^{p-1,q}\left(\mathcal{M}\times\mathcal{T}\right)$ by $\psi(z,t) := \tilde{\psi}(z,t,v(z,t),Dv(z,t))$ respectively, we may apply Theorem 5.3.2 to the equation (5.3.10). Thus we can exploit the Schauder estimates of Theorem 5.3.2 for equations in the quasi-linear form of (5.3.10). If $\tilde{\mathbf{a}}$ and $\tilde{\psi}$ are both independent of their fourth argument (i.e, do not depend on $Dv(z,t)$) then both $\mathbf{a}$ and ψ enjoy the same regularity as v.

Achieving specified Gauss curvature control will require good C^2 control on conformal factors; which itself will be obtained via interpolation. In particular, we require the following result from [GT11].

Lemma 5.3.3 (Lemma B.6 in [GT11])**.** *Let $\mathbb{B}^n(0,1) := \{x \in \mathbb{R}^n : |x| < 1\}$ denote the open unit ball in $\mathbb{R}^n$ and suppose $\phi : \mathbb{B}^n(0,1) \to [-1,1]$ is smooth and that for all $m \in \mathbb{N}$ we have $||D^m\phi||_{L^\infty(\mathbb{B}^n(0,1))} < \infty$. Then for all $k \in \mathbb{N}$ and $\eta \in (0,1)$ there exists a constant $C = C(k,\eta) > 0$ and $l := \left\lceil \frac{k}{\eta} \right\rceil$ such that*

$$|D^k\phi|(0) \leq C\left(1 + ||D^l\phi||_{L^\infty(\mathbb{B}^n(0,1))}\right) ||\phi||^{1-\eta}_{L^\infty(\mathbb{B}^n(0,1))}. \tag{5.3.11}$$

5.4. Hyperbolic Preservation Lemmata

Here we obtain a few lemmata recording how, and in what sense, various almost-hyperbolic conditions propagate forwards in time under Ricci flow. The first result establishes that if a flow $g(t)$ is initially locally almost-hyperbolic, then by reducing to a controllably smaller spatial region, the rescaled flow $\frac{g(t)}{1+2t}$ must remain close to being hyperbolic in a C^0 sense. The precise result is the following.

Lemma 5.4.1 (Barriers for rescaled flow; Lemma 3.1 in [McL18])**.** *There is a universal constant $\varepsilon > 0$ such that given any $b \in \left(0, \frac{1}{2}\right]$ there exists a constant $J = J(b) > 0$ for which the following holds:*

Assume that $R \geq J$ and $(\mathcal{M},\mathcal{H})$ is a smooth surface such that for some $x \in \mathcal{M}$ we have both $\mathbb{B}_{\mathcal{H}}(x,R) \subset\subset \mathcal{M}$ and that $(\mathbb{B}_{\mathcal{H}}(x,R),\mathcal{H})$ is isometric to a hyperbolic disc of radius R. Suppose $g(t)$ is a smooth Ricci flow defined on $\mathcal{M}$ for all $t \in [0,T]$, for some $T > 0$, with $g(0)$ conformal *to $\mathcal{H}$, and satisfying that for any $z \in \mathbb{B}_{\mathcal{H}}(x,R)$ and $t \in [0,T]$ we have $\mathbb{B}_{g(t)}(z,1) \subset\subset \mathcal{M}$. Further suppose that*

$$(i) \quad (1-b)\mathcal{H} \leq g(0) \leq (1+b)\mathcal{H} \qquad \textit{and} \qquad (ii) \quad \left|\mathrm{K}_{g(0)}\right| \leq 2 \tag{5.4.1}$$

throughout $\mathbb{B}_{\mathcal{H}}(x,R)$. Let $\tau := \min\{\varepsilon, T\} > 0$. Then we may conclude that

$$(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H} \tag{5.4.2}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-J) \times [0,\tau]$.

Observe that $\mathcal{H}_{\pm}(t) := (1 \pm b + 2t)\mathcal{H}$ are both Ricci flows with $\mathcal{H}_{+}(0) = (1+b)\mathcal{H}$ and $\mathcal{H}_{-}(0) = (1-b)\mathcal{H}$. Since $(1-b)\mathcal{H} < \frac{\mathcal{H}_{\pm}(t)}{1+2t} < (1+b)\mathcal{H}$ for positive times $t > 0$, it is reasonable to expect that on a smaller spatial region $g(t)$ should remain sandwiched as in (5.4.2) for a definite amount of time.

As we will see in the proof, the Gauss curvature bound assumed in (ii) of (5.4.1) means that Theorem 2.6.3 allows us to conclude that $(1-b)e^{-8t}\mathcal{H} \leq g(t) \leq (1+b)e^{8t}\mathcal{H}$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-2) \times [0,\varepsilon]$ for a universal $\varepsilon > 0$. By restricting ε to being sufficiently small, we see that this almost establishes (5.4.2) in that we can deduce that $\frac{g(t)}{1+9t} \leq (1+b)\mathcal{H}$ and $\frac{g(t)}{1-9t} \geq (1-b)\mathcal{H}$. The content of the lemma is to establish that we may replace $1+9t$ and $1-9t$ by the *same* function $1+2t$ and still preserve the barriers for a universal time $\varepsilon > 0$.

The improvement will follow from considering suitable dilations of the barrier flows $\mathcal{H}_{\pm}(t)$. Utilising barriers, in combination with the comparison principle, is a standard approach to two-dimensional Ricci flow; examples of which may be found in the works of Giesen and Topping [GT11, GT13], or Appendix C of the work [SSS10] of Schnürer, Schulze and Simon.

Proof of Lemma 5.4.1. Let h denote the complete conformal hyperbolic metric of constant Gauss curvature -1 on $\mathbb{D}$. Observe that $\mathrm{Vol}\mathbb{B}_h(z,r) \geq \pi r^2$ for all points $z \in \mathbb{D}$ and any radius $r \in (0,1]$. Let $\varepsilon > 0$ be the universal constant arising from appealing to the pseudolocality result of Chen, Theorem 2.6.3, with r_0 and v_0 there equal to $\frac{1}{\sqrt{2}}$ and $\frac{\pi}{4}$ respectively. In particular, this tells us that if $(M^2, g(t))$ is a smooth Ricci flow defined for $t \in [0,T]$, where $T > 0$ is arbitrary, and if $y \in M$ such that $\mathbb{B}_{g(t)}\left(y, \frac{1}{\sqrt{2}}\right) \subset\subset M$ for all $t \in [0,T]$, $|\mathrm{K}_{g(0)}| \leq 2$ throughout $\mathbb{B}_{g(0)}\left(y, \frac{1}{\sqrt{2}}\right)$ and $\mathrm{Vol}\mathbb{B}_{g(0)}\left(y, \frac{1}{\sqrt{2}}\right) \geq \frac{\pi}{8}$, then $|\mathrm{K}_{g(t)}(y)| \leq 4$ for all $t \in [0,\tau]$, where $\tau := \min\{\varepsilon, T\} > 0$. We fix this universal $\varepsilon > 0$ for the remainder of the proof.

Given $b \in \left(0, \frac{1}{2}\right]$ we seek to specify a constant $J = J(b) > 0$ so that, on a closed $\mathcal{H}$ ball of radius $R - J$, the barriers in (i) of (5.4.1) are valid for positive times for the rescaled family $\frac{g(t)}{1+2t}$.

With the benefit of hindsight, it will suffice to take

$$J(b) := 2 + \frac{1}{b}\max\left\{4e^{10\varepsilon}, 12\right\} > 2. \tag{5.4.3}$$

After locally pulling back to the disc $\mathbb{D}$, it will be convenient to work with the Euclidean distance. Recall from Section 5.2 that a h ball of radius r centred at $0 \in \mathbb{D}$ corresponds to a Euclidean ball of radius $\tanh(r/2)$ centred at 0. Later in the proof we will end up working on a h ball of radius $J-2$ centred at the origin $0 \in \mathbb{D}$, which corresponds to $\mathbb{D}_j$ where $j := \tanh((J-2)/2)$. For use later we record that the bounds in (5.4.3) give that

$$j := \tanh\left(\frac{J-2}{2}\right) \geq \max\left\{1 - \frac{b}{2}e^{-10\varepsilon}, 1 - \frac{b}{6}\right\} > 0 \tag{5.4.4}$$

via the inequality $\tanh(y) \geq 1 - \frac{1}{y}$ for $y > 0$ (cf. Lemma 5.2.1).

With both $\varepsilon > 0$ and $J > 0$ specified, we let $R \geq J, T > 0$ and define $\tau := \min\{\varepsilon, T\} > 0$. Assume that $g(t)$ is a smooth Ricci flow on $\mathcal{M}$, defined for all $t \in [0,T]$, with $g(0)$ conformal to $\mathcal{H}$, and satisfying that for every $z \in \mathbb{B}_{\mathcal{H}}(x,R)$ and every $t \in [0,T]$ we have $\mathbb{B}_{g(t)}(z,1) \subset\subset \mathcal{M}$. Further suppose $g(0)$ satisfies both estimates (i) and (ii) in (5.4.1) throughout $\mathbb{B}_{\mathcal{H}}(x,R)$.

Since $R \geq J > 2$ we may consider $z_0 \in \mathbb{B}_{\mathcal{H}}(x, R-3/2)$ so that $\mathbb{B}_{\mathcal{H}}(z_0,1) \subset\subset \mathbb{B}_{\mathcal{H}}(x,R)$. Moreover, the barrier estimates (i) of (5.4.1) ensure that

$$\mathbb{B}_{\mathcal{H}}\left(z_0, \frac{1}{2}\right) \subset \mathbb{B}_{g(0)}\left(z_0, \frac{\sqrt{3}}{2\sqrt{2}}\right) \subset \mathbb{B}_{g(0)}\left(z_0, \frac{1}{\sqrt{2}}\right) \subset \mathbb{B}_{\mathcal{H}}(z_0,1) \subset\subset \mathbb{B}_{\mathcal{H}}(x,R). \tag{5.4.5}$$

The inclusions of (5.4.5) allow us to simultaneously conclude that $|\mathrm{K}_{g(0)}| \leq 2$ throughout the ball $\mathbb{B}_{g(0)}\left(z_0, \frac{1}{\sqrt{2}}\right)$ via (ii) of (5.4.1), and that $\mathrm{Vol}\mathbb{B}_{g(0)}\left(z_0, \frac{1}{\sqrt{2}}\right) \geq \frac{\pi}{8}$. Recalling how $\varepsilon > 0$ was chosen, Theorem 2.6.3 tells us that $|\mathrm{K}_{g(t)}(z_0)| \leq 4$ for all $t \in [0,\tau]$. Repeating for all such points z_0 allows us to conclude that $|\mathrm{K}_{g(t)}| \leq 4$ throughout $\mathbb{B}_{\mathcal{H}}(x, R-3/2) \times [0,\tau]$. Recalling (5.2.1), estimate (i) in (5.4.1) and the Gauss curvature control allows us to conclude that $(1-b)e^{-8\varepsilon}\mathcal{H} \leq g(t) \leq (1+b)e^{8\varepsilon}\mathcal{H}$ throughout $\mathbb{B}_{\mathcal{H}}(x, R-3/2) \times [0,\tau]$.

To establish that $(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H}$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-J) \times [0,\tau]$ we pull back to $\mathbb{B}_h(0,R) \subset \mathbb{D}$. That is, we pull back via the isometry $F : (\mathbb{B}_h(0,R), h) \to (\mathbb{B}_{\mathcal{H}}(x,R), \mathcal{H})$. After doing so we have a smooth Ricci flow $F^*g(t)$ defined on $\mathbb{B}_h(0,R)$ for all $t \in [0,T]$, and in particular satisfying that $(1-b)h \leq F^*g(0) \leq (1+b)h$ throughout $\mathbb{B}_h(0,R)$ and $(1-b)e^{-8\varepsilon}h \leq F^*g(t) \leq (1+b)e^{8\varepsilon}h$ throughout $\mathbb{B}_h(0, R-3/2) \times [0,\tau]$. If we can establish that $(1-b)h \leq \frac{F^*g(t)}{1+2t} \leq (1+b)h$ throughout $\overline{\mathbb{B}}_h(0, R-J) \times [0,\tau]$ then the isometry will allow us to conclude (5.4.2) as required.

Given any $w \in \overline{\mathbb{B}}_h(0, R-J) \subset \mathbb{D}$ we can choose a Möbius diffeomorphism $\mathbb{D} \to \mathbb{D}$

mapping the origin 0 to w. Recalling from Section 5.2 that the pointwise difference between any metric and the hyperbolic metric h are preserved under pulling back via Möbius diffeomorphisms, establishing the following claim is sufficient to complete the proof.

Claim: Suppose $g(t)$ is a smooth Ricci flow on $\mathbb{B}_h(0, J-3/2)$, defined for all $t \in [0,\tau]$, and satisfying both $(1-b)h \leq g(0) \leq (1+b)h$ throughout $\mathbb{B}_h(0, J-3/2)$ and $(1-b)e^{-8\varepsilon}h \leq g(t) \leq (1+b)e^{8\varepsilon}h$ throughout $\mathbb{B}_h(0, J-3/2) \times [0,\tau]$. Then at the origin $0 \in \mathbb{D}$ we have $(1-b)h \leq \frac{g(t)}{1+2t} \leq (1+b)h$ for all $t \in [0,\tau]$.

Proof: Let $j_0 := \tanh\left(\frac{J-\frac{3}{2}}{2}\right)$ and recall that $j = \tanh\left(\frac{J-2}{2}\right)$ so that $\mathbb{B}_h(0, J-2) = \mathbb{D}_j \subset\subset \mathbb{D}_{j_0} = \mathbb{B}_h(0, J-3/2)$. Let $u : \mathbb{D}_{j_0} \times [0,\tau] \to \mathbb{R}$ be the smooth scalar function for which $g(t) = e^{2u}|dz|^2$. In particular, we have that $u \in C^\infty\left(\overline{\mathbb{D}_j} \times [0,\tau]\right)$. Recalling that $h = e^{2\varphi}|dz|^2$, where $\varphi(z) = \log\left[\frac{2}{1-|z|^2}\right]$, the barriers $(1-b)h \leq g(0) \leq (1+b)h$ and $(1-b)e^{-8\varepsilon}h \leq g(t) \leq (1+b)e^{8\varepsilon}h$ become

$$\frac{1}{2}\log(1-b) \leq u(z,0) - \varphi(z) \leq \frac{1}{2}\log(1+b) \tag{5.4.6}$$

for $z \in \mathbb{D}_{j_0}$, and

$$\frac{1}{2}\log(1-b) - 4\varepsilon \leq u(z,t) - \varphi(z) \leq \frac{1}{2}\log(1+b) + 4\varepsilon \tag{5.4.7}$$

for $(z,t) \in \mathbb{D}_{j_0} \times [0,\tau]$ respectively.

We now define suitable Ricci flows between which our flow $g(t)$ will remain sandwiched. The upper barrier will follow from considering a complete Ricci flow $h_\alpha(t)$ on the disc of radius $\alpha = \alpha(j) \in (j,1)$ with initial Gaussian curvature $-(1+b)^{-1}\alpha^{-2}$ where α is taken to be $\alpha(j) := \left(\frac{e^{4\varepsilon}j^2}{e^{4\varepsilon}+j^2-1}\right)^{\frac{1}{2}}$. By observing that $\alpha(s)$ is strictly increasing as a function of s and that $\alpha(0) = 0$ and $\alpha(1) = 1$ we see that $\alpha(j) \in (0,1)$. A simple computation verifies that $\alpha(j) > j$ as required. The conformal factor of this flow may be written as

$$H_\alpha(z,t) := \varphi_\alpha(z) + \frac{1}{2}\log(1+b) + \frac{1}{2}\log\left(1 + \frac{2t}{(1+b)\alpha^2}\right) \tag{5.4.8}$$

where $\varphi_\alpha(z) := \varphi\left(\frac{z}{\alpha}\right)$ so that $\varphi \leq \varphi_\alpha$ where both defined. In particular, one can compute from the definition of α that if $|z| = j$ then $\varphi_\alpha(z) = \varphi(z) + 4\varepsilon$ (having ensured $\alpha > j$ means that φ_α is defined for $|z| = j$).

As a function, $H_\alpha \in C^\infty(\mathbb{D}_\alpha \times [0,\infty))$ thus, in particular, smooth on $\overline{\mathbb{D}_j} \times [0,\tau]$ since $\mathbb{D}_j \subset\subset \mathbb{D}_\alpha$. Moreover, recalling (5.4.6), we see that (5.4.8) ensures that $H_\alpha(z,0) \geq u(z,0)$ throughout $\mathbb{D}_j$, whilst for $(z,t) \in \partial\mathbb{D}_j \times [0,\tau]$ we may compute, using (5.4.7), that $H_\alpha(z,t) \geq \varphi_\alpha(z) + \frac{1}{2}\log(1+b) = \varphi(z) + 4\varepsilon + \frac{1}{2}\log(1+b) \geq u(z,t)$ since $z \in \partial\mathbb{D}_j$ means $|z| = j$.

We are now in a position to apply the variant of the comparison principle stated in Theorem

5.2.3 to deduce that $H_\alpha \geq u$ throughout $\overline{\mathbb{D}_j} \times [0,\tau]$. Since at the origin $0 \in \mathbb{D}_j$ we have $\varphi_\alpha(0) = \varphi(0)$, we see that at the origin $H_\alpha \geq u$ is equivalent to

$$g(t) \leq \left(1 + b + \frac{2t}{\alpha^2}\right) h. \tag{5.4.9}$$

The lower barrier is constructed in a similar fashion. This time we consider a complete Ricci flow $h_\mu(t)$ on the disc of radius $\mu = \mu(j) > 1$ with Gaussian curvature initially $-(1-b)^{-1}\mu^{-2}$ where μ is taken to be $\mu(j) := j\left(1 - (1-j^2)\exp\left[\frac{5-4b}{1-b}\varepsilon\right]\right)^{-\frac{1}{2}}$. For this to make sense we require $1 - (1-j^2)\exp\left[\frac{5-4b}{1-b}\varepsilon\right] > 0$, which will be the case provided $1 - e^{-10\varepsilon} < j^2$. From (5.4.4) we know that $j \geq 1 - \frac{b}{2}e^{-10\varepsilon}$ and so, via Bernoulli's inequality, $j^2 > 1 - be^{-10\varepsilon}$ which is a little stronger than required. A straightforward computation shows that $\mu(j) > 1$ as claimed.

The restriction of this flow to $\mathbb{D}_j$ yields a (now incomplete) flow which acts as a lower barrier for our flow $g(t)$ on $\mathbb{D}_j$. To see this observe that the conformal factor of this flow can be written as

$$H_\mu(z,t) := \varphi_\mu(z) + \frac{1}{2}\log(1-b) + \frac{1}{2}\log\left(1 + \frac{2t}{(1-b)\mu^2}\right) \tag{5.4.10}$$

where $\varphi_\mu(z) := \varphi\left(\frac{z}{\mu}\right)$ so that $\varphi_\mu \leq \varphi$ where both defined. As a function $H_\mu \in C^\infty(\mathbb{D} \times [0,\infty))$ and thus, in particular, smooth on $\overline{\mathbb{D}_j} \times [0,\tau]$. Moreover, recalling (5.4.6), we see that (5.4.10) ensures that $H_\mu(z,0) \leq u(z,0)$ throughout $\mathbb{D}_j$. Further, if $z \in \partial\mathbb{D}_j$ then $|z| = j$ and so $\varphi_\mu(z) = \varphi(z) - 4\varepsilon - \frac{\varepsilon}{1-b}$. Therefore we may deduce that

$$\varphi_\mu(z) + \frac{1}{2}\log\left(1 + \frac{2t}{(1-b)\mu^2}\right) \leq \varphi_\mu(z) + \frac{t}{(1-b)\mu^2} \leq \varphi_\mu(z) + \frac{\varepsilon}{(1-b)} \leq \varphi(z) - 4\varepsilon \tag{5.4.11}$$

for all $(z,t) \in \partial\mathbb{D}_j \times [0,\tau]$ where we have used the inequality $\log x \leq x - 1$. Hence (5.4.7) and (5.4.11) allows us to conclude that $H_\mu \leq u$ throughout $\partial\mathbb{D}_j \times [0,\tau]$.

We are now in a position to apply the variant of the comparison principle stated in Theorem 5.2.3 to deduce that $H_\mu \leq u$ throughout $\overline{\mathbb{D}_j} \times [0,\tau]$. Since at the origin $0 \in \mathbb{D}_j$ we have $\varphi_\mu(0) = \varphi(0)$, we see that at the origin $H_\mu \leq u$ is equivalent to

$$\left(1 - b + \frac{2t}{\mu^2}\right) h \leq g(t). \tag{5.4.12}$$

Combining (5.4.9) and (5.4.12) yields that

$$(1-b)\left(\frac{1 + \frac{2t}{(1-b)\mu^2}}{1+2t}\right) h \leq \frac{g(t)}{1+2t} \leq (1+b)\left(\frac{1 + \frac{2t}{(1+b)\alpha^2}}{1+2t}\right) h \tag{5.4.13}$$

at the origin $0 \in \mathbb{D}$ for all times $t \in [0,\tau]$. The estimates of (5.4.13) yield the barriers required by the claim provided we have both

$$(\mathbf{A}) \quad \alpha^2 \geq \frac{1}{1+b} \qquad \text{and} \qquad (\mathbf{B}) \quad \mu^2 \leq \frac{1}{1-b}. \tag{5.4.14}$$

The estimate $(\mathbf{A})$ in (5.4.14) is true provided

$$j^2 \geq \frac{e^{4\varepsilon}-1}{e^{4\varepsilon}-1+be^{4\varepsilon}} = 1 - \frac{b}{1+b-e^{-4\varepsilon}}.$$

From (5.4.4) we know that $j \geq 1 - \frac{b}{6}$ and thus $j^2 \geq 1 - \frac{b}{3}$ via the Bernoulli inequality. This is a little stronger than required and hence $(\mathbf{A})$ in (5.4.14) is true. The estimate $(\mathbf{B})$ in (5.4.14) is true provided

$$j^2 \geq \frac{\exp\left[\frac{5-4b}{1-b}\varepsilon\right]-1}{\exp\left[\frac{5-4b}{1-b}\varepsilon\right]-1+b} = 1 - \frac{b}{\exp\left[\frac{5-4b}{1-b}\varepsilon\right]-1+b}.$$

From (5.4.4) we know that $j \geq 1 - \frac{b}{2}e^{-10\varepsilon}$ and thus $j^2 \geq 1 - be^{-10\varepsilon}$ via the Bernoulli inequality. This is stronger than required and hence $(\mathbf{B})$ in (5.4.14) is true. The estimates $(\mathbf{A})$ and $(\mathbf{B})$ in (5.4.14) combine with (5.4.13) to yield that $(1-b)h \leq \frac{g(t)}{1+2t} \leq (1+b)h$ at the origin $0 \in \mathbb{D}$ for all $t \in [0,\tau]$, thus completing the proof of the claim. ††

Combined with suitable Möbius diffeomorphisms, the claim allows us to establish the desired barriers for the pulled back flow $F^*g(t)$ on $\overline{\mathbb{B}}_h(0, R-J) \times [0,\tau]$. The barriers in (5.4.2) on $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-J) \times [0,\tau]$ are then immediate by pulling back via the diffeomorphism F^{-1}. This completes the proof of Lemma 5.4.1. ■

It is well known that L^∞ barriers give rise to uniform C^k estimates at strictly positive times; examples of this may be found in [Gie12] or Appendix C of [SSS10], say. The following result uses this to establish Gauss curvature control away from time 0.

Lemma 5.4.2 (Barriers give curvature control; Lemma 3.2 in [McL18])**.** *Let $\alpha \in (0,1]$ and $S > 0$. Then for any $\delta \in (0,S)$ there exists a constant $b = b(S,\alpha,\delta) > 0$ for which the following is true.*

Assume that $(\mathcal{M},\mathcal{H})$ is a smooth surface such that for some $x \in \mathcal{M}$ and $R \geq 2$ we have $\mathbb{B}_{\mathcal{H}}(x,R) \subset\subset \mathcal{M}$ and that $(\mathbb{B}_{\mathcal{H}}(x,R),\mathcal{H})$ is isometric to a hyperbolic disc of radius R. Suppose that $g(t)$ is a smooth Ricci flow on $\mathcal{M}$, defined for all $t \in [0,T]$ for some $T \in (0,S]$, with $g(0)$ conformal to $\mathcal{H}$, and we have the barriers

$$(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H} \tag{5.4.15}$$

throughout $\mathbb{B}_{\mathcal{H}}(x,R)\times[0,T]$. *Then we may conclude that we have the Gauss curvature bounds*

$$-1-\alpha\leq \mathrm{K}_{\frac{g(t)}{1+2t}}\leq -1+\alpha \tag{5.4.16}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-2)\times[\delta,T]$. *The estimates of* (5.4.16) *are vacuous if* $T<\delta$.

Proof of Lemma 5.4.2. We briefly summarise the strategy of the proof. Initially we pull back to the disc $\mathbb{D}$ via the isometry $F:(\mathbb{B}_h(0,R),h)\to(\mathbb{B}_{\mathcal{H}}(x,R),\mathcal{H})$, which we know exists by assumption. Once pulled back, we obtain the result for the flow $F^*g(t)$ and then conclude the desired estimates for $g(t)$ itself by pulling back via F^{-1}. Before doing so, we first establish the following claim which will be of use after we have pulled back to the disc $\mathbb{D}$.

Claim: Let $a\in(0,1)$ and $0<\delta\leq S$. There exists a constant $\mathcal{N}=\mathcal{N}(S,a,\delta)\in(0,\infty)$ for which the following is true.

Assume that $g(t)$ is a smooth Ricci flow on $\mathbb{B}_h(0,2)$, defined for all $t\in[0,T]$ for some $T\in(0,S]$, with $g(t)=wh$ for some smooth function $w:\mathbb{B}_h(0,2)\times[0,T]\to\mathbb{R}$, and satisfying, for some $b\in[0,a]$, that $(1-b)h\leq\frac{g(t)}{1+2t}\leq(1+b)h$ throughout $\mathbb{B}_h(0,2)\times[0,T]$. Then at the origin 0 we have, for any $t\in[\delta,T]$, the Gauss curvature estimates

$$-\mathcal{N}b^{\frac{1}{3}}\leq \mathrm{K}_{\frac{g(t)}{1+2t}}(0)-\mathrm{K}_h(0)\leq\mathcal{N}b^{\frac{1}{3}}. \tag{5.4.17}$$

Proof: Since (5.4.17) is vacuous for $T<\delta$ we need only deal with the case when $T\geq\delta$. Let $a\in(0,1)$ and $0<\delta\leq T\leq S$ all be given. Let $\varphi:\mathbb{D}\to\mathbb{R}$ be the smooth function for which the complete conformal hyperbolic metric of constant Gauss curvature -1 on $\mathbb{D}$ is given by $h=e^{2\varphi}|dz|^2$. Since φ is smooth we can choose a sequence $E_l>0$, defined for $l\in\mathbb{N}$, for which $||\varphi||_{C^l(\mathbb{D}_{1/2};g_E)}\leq E_l$, where g_E denotes the flat Euclidean metric on $\mathbb{D}$. Using this estimate for $l=1$ allows us to choose a constant $K=K(a)>0$ such that if $v:\mathbb{D}_{1/2}\to\mathbb{R}$ is smooth and $|v-\varphi|\leq\frac{a}{1-a}$ throughout $\mathbb{D}_{1/2}$ then $|v|\leq K$ throughout $\mathbb{D}_{1/2}$.

Now let $g(t)$ be a smooth Ricci flow satisfying the hypotheses of the claim. Observe that $\mathbb{D}_{1/2}=\mathbb{B}_h(0,\log 3)\subset\subset\mathbb{B}_h(0,2)$ and thus $g(t)$ is defined throughout $\mathbb{D}_{1/2}\times[0,T]$. Moreover, if we let $u:\mathbb{D}_{1/2}\times[0,T]\to\mathbb{R}$ be a smooth conformal factor for which $g(t)=e^{2u}|dz|^2$, then the barriers $(1-b)h\leq\frac{g(t)}{1+2t}\leq(1+b)h$ throughout $\mathbb{B}_h(0,2)\times[0,T]$ yield that

$$|u-\frac{1}{2}\log(1+2t)-\varphi|\leq\frac{1}{2}\log\left(\frac{1}{1-b}\right)\leq\frac{b}{1-b}\leq\frac{b}{1-a}\leq\frac{a}{1-a} \tag{5.4.18}$$

throughout $\mathbb{D}_{1/2}\times[0,T]$, where we have used that $\log(1+x)\leq x$.

Recalling how $K>0$ was chosen above, and that $0<T\leq S$, (5.4.18) allows us to deduce

that

$$|u| \leq K + \frac{1}{2}\log(1+2S) \leq K + S =: A(S,a) \in (0,\infty). \tag{5.4.19}$$

Moreover, by virtue of $g(t)$ being a smooth Ricci flow we know, recall (5.2.2), that $u : \mathbb{D}_{1/2} \times [0,T] \to \mathbb{R}$ is a smooth solution to

$$\frac{\partial u}{\partial t} = e^{-2u}\Delta u. \tag{5.4.20}$$

The evolution equation (5.4.20) may be rewritten in the forms required by the regularity results Theorem 5.3.1 and Theorem 5.3.2. First we may rewrite it as

$$\frac{\partial u}{\partial t} = \operatorname{div}[\mathbf{A}(z,t,u(z,t),Du(z,t))] + B(z,t,u(z,t),Du(z,t)) \tag{5.4.21}$$

where $\mathbf{A} : \mathbb{D}_{1/2}\times[0,T]\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ and $B : \mathbb{D}_{1/2}\times[0,T]\times\mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ are given respectively by $\mathbf{A}(z,t,w,\xi) := e^{-2w}\xi$ and $B(z,t,w,\xi) := 2e^{-2w}|\xi|^2$. Taking $\lambda(\theta) := e^{-2\theta}$, $\Lambda(\theta) := e^{2\theta}$ and $\beta(\theta) := 2e^{2\theta}$, and recalling (5.4.19), we see that for all $(z,t,w,\xi) \in \mathbb{D}_{1/2} \times [0,T] \times \mathbb{R} \times \mathbb{R}^2$ we have

1. $\langle \mathbf{A}(z,t,w,\xi), \xi\rangle = e^{-2w}|\xi|^2 \geq \lambda(|w|)|\xi|^2$,
2. $|\mathbf{A}(z,t,w,\xi)| \leq e^{2|w|}|\xi| = \Lambda(|w|)|\xi|$, and
3. $|\mathbf{B}(z,t,w,\xi)| \leq 2e^{2|w|}|\xi|^2 = \beta(|w|)|\xi|^2$.

Together, (5.4.19) and $1-3$ above yield that Theorem 5.3.1 may be applied to the evolution equation in (5.4.21).

Secondly we may write (5.4.20) as

$$\frac{\partial u}{\partial t}(z,t) = \langle \tilde{\mathbf{a}}(z,t,u(z,t)), \operatorname{Hess}(u)(z,t)\rangle \tag{5.4.22}$$

where $\tilde{\mathbf{a}} : \mathbb{D}_{1/2} \times [0,T] \times \mathbb{R} \to \operatorname{Sym}_2 \mathbb{R}^2$ is given by $\tilde{\mathbf{a}}(z,t,w) := e^{-2w}\operatorname{id}$. Motivated by the discussion after (5.3.10), we define $\mathbf{a}(z,t) := \tilde{\mathbf{a}}(z,t,u(z,t))$ and observe that the evolution equation in (5.4.22) is of the form required by Theorem 5.3.2. Further, recalling the estimate (5.4.19), we observe that for all $\xi \in \mathbb{R}^2 \setminus \{0\}$ we have that $\langle \mathbf{a}(z,t), \xi\otimes\xi\rangle \geq \rho|\xi|^2 > 0$, where $\rho := e^{-2A} > 0$ for the constant A arising in (5.4.19). Finally we can conclude that $\mathbf{a}$ enjoys the same regularity as u, and even that

$$||\mathbf{a}||_{C^{r_1+\gamma,r_2+\kappa}(\cdot;\operatorname{Sym}_2\mathbb{R}^2)} \leq P\left(r_1,r_2,\gamma,\kappa,||u||_{C^{r_1+\gamma,r_2+\kappa}(\cdot)}\right) \tag{5.4.23}$$

for any $r_1, r_2 \in \mathbb{N}$, $\gamma,\kappa \in (0,1]$ and with the norms being taken over the same region. Altogether, we have established that Theorem 5.3.2 may be applied to the evolution equation in (5.4.22).

From (5.4.19) we have that $|u| \leq A$ throughout $\mathbb{D}_{1/2} \times [0,T]$. Hence the evolution equation in (5.4.21) is uniformly parabolic on $\mathbb{D}_{1/2} \times [0,T]$, and of the required form in order to appeal to Theorem 5.3.1 to deduce Hölder estimates for u. In turn, these Hölder bounds provide the required regularity to appeal to parabolic Schauder estimates (Theorem 5.3.2), with (5.4.23) ensuring $\mathbf{a}$ has the required Hölder regularity, to bootstrap, and obtain, for any $l \in \mathbb{N}_0$, constants $\tilde{C} = \tilde{C}(l,S,a,\delta) > 0$ for which

$$||u||_{C^{2l+\alpha,l+\frac{\alpha}{2}}(\mathbb{D}_{1/4}\times[\delta,T])} \leq \tilde{C} \tag{5.4.24}$$

where $\alpha = \alpha(a) \in (0,1]$.

A particular consequence of (5.4.24) is that we obtain a sequence $K_l > 0$, defined for $l \in \mathbb{N}$, depending only on a, δ and S such that for all $t \in [\delta,T]$ we have

$$||u(t)||_{C^l(\mathbb{D}_{1/4};g_E)} \leq K_l. \tag{5.4.25}$$

Moreover, recalling that $0 \leq \frac{1}{2}\log(1+2S) \leq S$, we may take the sequence $C_l := E_l + K_l + S > 0$, for $l \in \mathbb{N}$, depending only on a, δ and S for which for all $t \in [\delta,T]$ we have

$$||v(t) - \varphi||_{C^l(\mathbb{D}_{1/4};g_E)} \leq C_l \tag{5.4.26}$$

where $v(t) := u(t) - \frac{1}{2}\log(1+2t)$. To establish our desired Gauss curvature control, we interpolate between the C^0 control given by (5.4.18) and the C^3 control given by (5.4.26) (for $l = 3$) to obtain improved C^2 control on $v(t) - \varphi$ than the estimate provided by (5.4.26) for $l = 2$.

We will interpolate via Lemma 5.3.3. In order to appeal to this result consider a radially symmetric non-increasing smooth cut-off function $\theta \in C_c^\infty(\mathbb{D})$ for which

$$\theta(z) = \begin{cases} 1 & \text{if } z \in \mathbb{D}_{1/16} \\ 0 & \text{if } z \in \mathbb{D} \setminus \mathbb{D}_{3/16} \end{cases} \tag{5.4.27}$$

and with $||\theta||_{C^3(\mathbb{D};g_E)} \leq \hat{C}$ for some universal $\hat{C} > 0$. For a fixed $t \in [\delta,T]$ consider

$$F(z) := (1-a)\theta(z)\left(v(z,t) - \varphi(z)\right) \tag{5.4.28}$$

which is a smooth map $\mathbb{D} \to \mathbb{R}$ and satisfies, recalling (5.4.18), that $|F| \leq 1$ throughout $\mathbb{D}$. Moreover, recalling (5.4.26) for $l = 3$, a simple computation yields that $||F||_{C^3(\mathbb{D};g_E)} \leq Z$ for some $Z = Z(S,a,\delta) > 0$.

Thus we could appeal to Lemma 5.3.3 to control $|D^2F|_{g_E}$ at the origin. However, we want to control $|\nabla_h^2 F|_h$ at the origin, and to achieve this using Lemma 5.2.2 requires control on the derivatives, with respect to the Euclidean metric g_E, up to second order.Therefore we will need to interpolate via Lemma 5.3.3 twice. In the first case we apply Lemma 5.3.3 to F, with $k = 1$ and

$\eta = \frac{1}{2}$, to obtain that

$$|DF|_{g_E}(0) \leq \mathcal{B}_1 \left(1 + ||D^2F||_{L^\infty(\mathbb{D};g_E)}\right) ||F||^{\frac{1}{2}}_{L^\infty(\mathbb{D})} \leq \mathcal{B}_1(1+Z)||F||^{\frac{1}{2}}_{L^\infty(\mathbb{D})} \tag{5.4.29}$$

for a universal $\mathcal{B}_1 > 0$. In the second case we apply Lemma 5.3.3 to F, with $k = 2$ and $\eta = \frac{2}{3}$, to deduce that

$$|D^2F|_{g_E}(0) \leq \mathcal{B}_2 \left(1 + ||D^3F||_{L^\infty(\mathbb{D};g_E)}\right) ||F||^{\frac{1}{3}}_{L^\infty(\mathbb{D})} \leq \mathcal{B}_2(1+Z)||F||^{\frac{1}{3}}_{L^\infty(\mathbb{D})} \tag{5.4.30}$$

for a universal $\mathcal{B}_2 > 0$.

From (5.4.27), we have $DF(0) = (1-a)D(v(0,t)-\varphi(0))$, and $D^2F(0) = (1-a)D^2(v(0,t)-\varphi(0))$. Further $||F||_{L^\infty(\mathbb{D})} \leq (1-a)||v(t)-\varphi||_{L^\infty(\mathbb{D}_{1/4})}$, which is immediate from (5.4.28), and hence (5.4.29) becomes

$$|D(v(t)-\varphi)|_{g_E}(0) \leq \mathcal{B}_1(1+Z)(1-a)^{-\frac{1}{2}}||v(t)-\varphi||^{\frac{1}{2}}_{L^\infty(\mathbb{D}_{1/4})}, \tag{5.4.31}$$

whilst (5.4.30) becomes

$$|D^2(v(t)-\varphi)|_{g_E}(0) \leq \mathcal{B}_2(1+Z)(1-a)^{-\frac{2}{3}}||v(t)-\varphi||^{\frac{1}{3}}_{L^\infty(\mathbb{D}_{1/4})}. \tag{5.4.32}$$

We now appeal to the second part of Lemma 5.2.2, i.e. to (5.2.6). The result is that, for a universal $c > 0$, we have that

$$|\nabla^2_h(v(t)-\varphi)|_h(0) \leq c\sum_{k=0}^{2}|D^k(v(t)-\varphi)|_{g_E}(0) \tag{5.4.33}$$

Combining (5.4.18), (5.4.31), (5.4.32) and (5.4.33) yields that

$$|\nabla^2_h(v(t)-\varphi)|_h(0) \leq Q||v(t)-\varphi||^{\frac{1}{3}}_{L^\infty(\mathbb{D}_{1/4})} \tag{5.4.34}$$

where $Q = Q(S,a,\delta) := c(1-a)^{-\frac{2}{3}}\left(a^{\frac{2}{3}} + \mathcal{B}_1(1+Z)a^{\frac{1}{6}} + \mathcal{B}_2(1+Z)\right) > 0$. The arbitrariness of $t \in [\delta, T]$ allows us to conclude that (5.4.34) is valid for all $t \in [\delta, T]$. Finally we define $\mathcal{N} = \mathcal{N}(S,a,\delta) := \frac{1+2Q}{(1-a)^2} > 0$.

Recall from (5.2.3) that, for any $t \in [\delta, T]$, we have

$$\mathrm{K}_{\frac{g(t)}{1+2t}} - \mathrm{K}_h = -e^{-2(v(0,t)-\varphi(0))}\Delta_h(v-\varphi)(0) + (1-e^{-2(v(0,t)-\varphi(0))}). \tag{5.4.35}$$

From (5.4.18), (5.4.34) and that $0 < a < 1$ we deduce that

$$|\Delta_h(v-\varphi)(0)| \leq 2|\nabla_h^2(v(t)-\varphi)|_h(0) \leq 2Q(1-a)^{-\frac{1}{3}}b^{\frac{1}{3}} \leq \frac{2Qb^{\frac{1}{3}}}{1-a} \tag{5.4.36}$$

Therefore, for fixed $t \in [\delta, T]$, we can first estimate that

$$\begin{aligned}
\mathrm{K}_{\frac{g(t)}{1+2t}}(0) - \mathrm{K}_h(0) &\leq e^{2|v-\varphi|(0)}|\Delta_h(v-\varphi)|(0) + 1 - e^{-2|v(0)-\varphi(0)|} \\
&\leq e^{2|v-\varphi|(0)}|\Delta_h(v-\varphi)|(0) + 2|v(0)-\varphi(0)| \\
&\overset{(5.4.36)}{\leq} \frac{2Qb^{\frac{1}{3}}}{1-a}e^{2|v(0)-\varphi(0)|} + 2|v(0)-\varphi(0)| \\
&\overset{(5.4.18)}{\leq} \frac{2Qb^{\frac{1}{3}}}{(1-b)(1-a)} + \frac{b}{1-b} \leq \frac{1+2Q}{(1-a)^2}b^{\frac{1}{3}} = \mathcal{N}b^{\frac{1}{3}}
\end{aligned}$$

using the inequality $e^x \geq 1 + x$ in the second line and recalling that $0 \leq b \leq a < 1$ in the last line. Similarly

$$\begin{aligned}
\mathrm{K}_{\frac{g(t)}{1+2t}}(0) - \mathrm{K}_h(0) &\geq -e^{2|v-\varphi|(0)}|\Delta_h(v-\varphi)|(0) + 1 - e^{2|v(0)-\varphi(0)|} \\
&\overset{(5.4.18)}{\geq} -\frac{2}{1-b}|\nabla_h^2(\tilde{u}-\varphi)|_h(0) + 1 - 1 - \frac{b}{1-b} \\
&\overset{(5.4.36)}{\geq} -\frac{2Qb^{\frac{1}{3}}}{(1-b)(1-a)} - \frac{b}{1-b} \geq \frac{1+2Q}{(1-a)^2}b^{\frac{1}{3}} = -\mathcal{N}b^{\frac{1}{3}}
\end{aligned}$$

where again we have used $0 \leq b \leq a < 1$. Since $t \in [\delta, T]$ was chosen arbitrarily, we may conclude that $-\mathcal{N}b^{\frac{1}{3}} \leq \mathrm{K}_{\frac{g(t)}{1+2t}}(0) - \mathrm{K}_h(0) \leq \mathcal{N}b^{\frac{1}{3}}$ for all $t \in [\delta, T]$, as required in (5.4.17). ††

With the claim established we may now prove Lemma 5.4.2. Let $\mathcal{N} = \mathcal{N}(S, \delta) > 0$ be the constant given by the claim for the S and δ as in the statement of Lemma 5.4.2 and $a \in (0, 1)$ given by $1/2$. Then define $b = b(S, \alpha, \delta) := \left(\frac{\alpha}{\mathcal{N}}\right)^3 > 0$. If necessary we may reduce b, without additional dependency, to ensure $b \in (0, 1/2]$. With the constant b specified, we need only verify that the claimed assertion is valid.

For this purpose assume that $(\mathcal{M}, \mathcal{H})$ is a smooth surface as specified in the statement of the lemma. Moreover, let $g(t)$ be a smooth Ricci flow defined on $\mathcal{M}$ for all times $t \in [0, T]$, conformally equivalent to $\mathcal{H}$, and satisfying the barriers specified in (5.4.15) throughout $\mathbb{B}_{\mathcal{H}}(x, R) \times [0, T]$. Let $F : (\mathbb{B}_h(0, R), h) \to (\mathbb{B}_{\mathcal{H}}(x, R), \mathcal{H})$ be an isometry, which exists since $(\mathbb{B}_{\mathcal{H}}(x, R), \mathcal{H})$ is isometric to a hyperbolic disc of radius R by assumption. We can consider the pull back $F^*g(t)$ which is a smooth Ricci flow defined on $\mathbb{B}_h(0, R)$ for all $t \in [0, T]$. Further, the barrier estimates in (5.4.15) throughout $\mathbb{B}_{\mathcal{H}}(x, R) \times [0, T]$ yield that, after being pulled back, the flow $F^*g(t)$ satisfies

$$(1-b)h \leq \frac{F^*g(t)}{1+2t} \leq (1+b)h \tag{5.4.37}$$

throughout $\mathbb{B}_h(0,R) \times [0,T]$. Moreover, since $g(0)$ is conformal to $\mathcal{H}$ we will also have that $F^*g(t) = vh$ for some smooth function $v : \mathbb{B}_h(0,R) \times [0,T] \to \mathbb{R}$.

Given any $w \in \overline{\mathbb{B}}_h(0,R-2)$, where $\overline{\mathbb{B}}_h(0,R-2)$ denotes the closed ball, we can choose a Möbius diffeomorphism $\psi_w : \mathbb{D} \to \mathbb{D}$ mapping the origin 0 to w. Recalling from Section 5.2 that the barriers in (5.4.37) are preserved under the pull back of ψ_w. That is, the pulled back flow $\psi_w^* F^* g(t)$ will now satisfy $(1-b)h \leq \frac{\psi_w^* F^* g(t)}{1+2t} \leq (1+b)h$ throughout $\mathbb{B}_h(0,2) \times [0,T]$. This allows us to apply the claim to deduce that at the origin 0 we have $-\mathcal{N} b^{\frac{1}{3}} \leq \mathrm{K}_{\frac{\psi_w^* F^* g(t)}{1+2t}} - \mathrm{K}_h \leq \mathcal{N} b^{\frac{1}{3}}$ for all $t \in [\delta, T]$. Recall, see Section 5.2, the pointwise difference $\mathrm{K}_{\frac{\psi_w^* F^* g(t)}{1+2t}} - \mathrm{K}_h$ is preserved under pull back by Möbius diffeomorphisms. Hence, pulling back by the Möbius diffeomorphism ψ_w^{-1} allows us to conclude that $-\mathcal{N} b^{\frac{1}{3}} \leq \mathrm{K}_{\frac{F^* g(t)}{1+2t}}(w) - \mathrm{K}_h(w) \leq \mathcal{N} b^{\frac{1}{3}}$ for all times $t \in [\delta, T]$.

The arbitrariness of $w \in \overline{\mathbb{B}}_h(0,R-2)$ allows us to conclude that $-\mathcal{N} b^{\frac{1}{3}} \leq \mathrm{K}_{\frac{F^* g(t)}{1+2t}} - \mathrm{K}_h \leq \mathcal{N} b^{\frac{1}{3}}$ throughout $\overline{\mathbb{B}}_h(0,R-2) \times [\delta,T]$. Since $\mathrm{K}_h \equiv -1$ and, by our choice of b, we have $\mathcal{N} b^{\frac{1}{3}} = \alpha$, we have in fact established that $-1-\alpha \leq \mathrm{K}_{\frac{F^* g(t)}{1+2t}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_h(0,R-2) \times [\delta,T]$. Pulling back via F^{-1} yields that $-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-2) \times [\delta,T]$ as required in (5.4.16), thus completing the proof of Lemma 5.4.2. ∎

At the very end of our overarching theorem, Theorem 5.5.1, in the case that we assume $g(0) \equiv \mathcal{H}$ throughout $\mathcal{M}$, we will require a slight modification of Lemma 5.4.2 to avoid any time delay before achieving our desired Gauss curvature control. The result will exploit the uniform initial C^l bounds provided by the initial equality $g(0) \equiv \mathcal{H}$.

Lemma 5.4.3 (No time delay; Lemma 3.3 in [McL18])**.** *Let $\alpha \in (0,1]$ and $S > 0$. Then there exists a constant $b = b(S,\alpha) > 0$ for which the following is true.*

Assume $(\mathcal{M},\mathcal{H})$ is a smooth surface such that for some $x \in \mathcal{M}$ and $R \geq 2$ we have $\mathbb{B}_{\mathcal{H}}(x,R) \subset\subset \mathcal{M}$ and $(\mathbb{B}_{\mathcal{H}}(x,R),\mathcal{H})$ is isometric to a hyperbolic disc of radius R. Suppose $g(t)$ is a smooth Ricci flow on $\mathcal{M}$, defined for all $t \in [0,T]$ for some $T \in (0,S]$, with $g(0) \equiv \mathcal{H}$ throughout $\mathcal{M}$, and we have the barriers

$$(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H} \tag{5.4.38}$$

throughout $\mathbb{B}_{\mathcal{H}}(x,R) \times [0,T]$. Then we may deduce that we have the Gauss curvature bounds

$$-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}} \leq -1+\alpha \tag{5.4.39}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-2) \times [0,T]$.

Proof of Lemma 5.4.3. As in the proof of Lemma 5.4.2 our strategy is to pull back to the disc

$\mathbb{D}$ via the isometry $F : (\mathbb{B}_h(0,R), h) \to (\mathbb{B}_{\mathcal{H}}(x,R), \mathcal{H})$, which we know exists by assumption, obtain the result for the pulled back flow $F^*g(t)$ and then conclude the desired estimates for $g(t)$ itself by pulling back via F^{-1}. Before doing so, we first establish the following claim which will be of use after we have pulled back to the disc $\mathbb{D}$.

Claim: Let $a \in (0,1)$ and $0 < T \leq S$. Then there exists a constant $\mathcal{N} = \mathcal{N}(S,a) \in (0,\infty)$ for which the following is true.

Assume $g(t)$ is a smooth Ricci flow on $\mathbb{B}_h(0,2)$, defined for all $t \in [0,T]$, with both $g(0) \equiv h$ throughout $\mathbb{B}_h(0,2)$ and $g(t) = wh$ for some smooth function $w : \mathbb{B}_h(0,2) \times [0,T] \to \mathbb{R}$, and satisfying, for some $b \in [0,a]$, the barriers $(1-b)h \leq \frac{g(t)}{1+2t} \leq (1+b)h$ throughout $\mathbb{B}_h(0,2) \times [0,T]$. Then at the origin 0 we have, for all times $t \in [0,T]$, the Gauss curvature estimates

$$-\mathcal{N}b^{\frac{1}{3}} \leq \mathrm{K}_{\frac{g(t)}{1+2t}}(0) - \mathrm{K}_h(0) \leq \mathcal{N}b^{\frac{1}{3}}. \tag{5.4.40}$$

Proof: To prove the claim, we proceed identically to how we established the corresponding claim within the proof of Lemma 5.4.2. The only difference is that the initial equality now provides time $t = 0$ C^l estimates for the conformal factor u such that $g(t) = e^{2u}|dz|^2$. These estimates mean that when we come to appeal to Theorems 5.3.1 and 5.3.2, which may be applied due to the same reasoning as in the proof of Lemma 5.4.2, we are now able to use the variants that only require moving away from the spatial boundary, as opposed to the entire parabolic boundary. As a result, the C^l estimates achieved for u in (5.4.25) may now be assumed to be valid at all times $t \in [0,T]$. From here we proceed verbatim to the proof of the claim in the proof of Lemma 5.4.2, recalling that the interpolation argument there established the Gauss curvature bounds at any time t for which the C^l estimates in (5.4.26) were valid. Therefore, since such estimates are now valid for all times $t \in [0,T]$, we now establish the Gauss curvature estimates (5.4.40) for all times $t \in [0,T]$ as required. ††

With this claim established, we follow the proof of Lemma 5.4.2 verbatim, differing only by using the claim here in place of the claim obtained within the proof of Lemma 5.4.2. ■

5.5. Improved Time Control

The following theorem will give both Theorem 5.1.1 and Theorem 5.1.6 as consequences.

Theorem 5.5.1 (Theorem 4.1 in [McL18])**.** *Let $\alpha \in (0,1]$ be given. Then there is a universal constant $\varepsilon > 0$ such that for any $\delta \in (0,\varepsilon)$ there exist constants $b = b(\alpha,\delta) > 0$ and $\Lambda = \Lambda(\alpha,\delta) > 0$ for which the following is true.*

Suppose that $R \geq \Lambda$ and that $(\mathcal{M}, \mathcal{H})$ is a smooth surface which satisfies for some $x \in \mathcal{M}$ that the ball $\mathbb{B}_{\mathcal{H}}(x, R) \subset\subset \mathcal{M}$ and $(\mathbb{B}_{\mathcal{H}}(x, R), \mathcal{H})$ is isometric to a hyperbolic disc of radius R. Assume $g(t)$ is a smooth Ricci flow defined on $\mathcal{M}$ for all $t \in [0, T]$ for some $T > 0$, with $g(0)$ conformal to $\mathcal{H}$, and satisfying that for any $l \in \mathbb{N}_0$, if $z \in \mathbb{B}_{\mathcal{H}}(x, R - l\Lambda)$ and $t \in [0, T]$ then $\mathbb{B}_{g(t)}\left(z, (1+2\varepsilon)^{\frac{l}{2}}\right) \subset\subset \mathcal{M}$. Further suppose that

$$(\mathbf{A}) \quad (1-b)\mathcal{H} \leq g(0) \leq (1+b)\mathcal{H} \qquad \textit{and} \qquad (\mathbf{B}) \quad |\mathrm{K}_{g(0)}| \leq 2 \tag{5.5.1}$$

throughout $\mathbb{B}_{\mathcal{H}}(x, R)$. Then we have that

$$-1 - \alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}} \leq -1 + \alpha \tag{5.5.2}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \lfloor \frac{R}{\Lambda} \rfloor \Lambda\right) \times [\delta, \mathcal{T}_{\max}]$ where

$$\mathcal{T}_{\max} = \min\left\{T, \frac{\exp\left[\lfloor \frac{R}{\Lambda} \rfloor \log(1+2\varepsilon)\right] - 1}{2}\right\}. \tag{5.5.3}$$

Moreover, if in place of the estimates in (5.5.1) we had that $g(0) \equiv \mathcal{H}$ throughout $\mathcal{M}$, then we may in fact deduce the estimates of (5.5.2) throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \lfloor \frac{R}{\Lambda} \rfloor \Lambda\right) \times [0, \mathcal{T}_{\max}]$, where $\mathcal{T}_{\max}$ is as specified in (5.5.3).

To clarify, for $z \in \mathbb{R}$ we have $\lfloor z \rfloor := \max\{m \in \mathbb{Z} : m \leq z\}$.

Proof of Theorem 5.5.1. Retrieve the universal constant $\varepsilon > 0$ from Lemma 5.4.1. Let $\alpha \in (0, 1]$ and $\delta \in (0, \varepsilon)$ both be given. Retrieve the constant $b_1 = b_1(\alpha, \delta) > 0$ arising in Lemma 5.4.2 for the S, α and δ there equal to ε, α and δ here respectively. With the aim of avoiding any time delay before obtaining the estimates of (5.5.2) in the case $g(0) \equiv \mathcal{H}$, retrieve the constant $b_2 = b_2(\alpha) > 0$ arising in Lemma 5.4.3 for the S and α there given by ε and α here respectively. Take $b := \min\{b_1, b_2\} > 0$ which depends only on α and δ. By reducing b if required, but without additional dependency, we may assume that $b \in (0, 1/2]$. This means we may define $\Lambda = \Lambda(\alpha, \delta) := J(b) + 2 > 0$ where $J(b)$ is the constant arising in Lemma 5.4.1. We fix these quantities for the remainder of the proof.

We first deal with the case $T \in (0, \varepsilon]$. That is, assume we are in the setting of the theorem with $T \in (0, \varepsilon]$. The estimates on $g(0)$ in (5.5.1), together with the assumed compact inclusions for $l = 0$ and that $g(0)$ is conformal to $\mathcal{H}$, provide the required hypotheses to apply Lemma 5.4.1 to the flow $g(t)$. Doing so, and recalling that $\tau := \min\{T, \varepsilon\} = T \leq \varepsilon$, yields the barriers $(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H}$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-\Lambda+2) \times [0, T]$, recalling that $\Lambda = J(b)+2 > 0$ where $J(b)$ is the constant arising in Lemma 5.4.1.

In turn, these barriers are of the form required by Lemma 5.4.2. Recalling how b was specified, we observe that we have the required hypothesis to apply Lemma 5.4.2 to $g(t)$ and deduce that $-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-\Lambda) \times [\delta, T]$. Of course, these Gauss curvature estimates are vacuous if $T < \delta$. Since $R \geq \Lambda$ we see that $\left\lfloor \frac{R}{\Lambda} \right\rfloor \geq 1$, and so we have established the Gauss curvature estimates required in (5.5.2) throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda\right) \times [\delta, T]$, which is for the time required in (5.5.3).

In the case that the estimates in (5.5.1) are replaced by the assumption that $g(0) \equiv \mathcal{H}$ throughout $\mathcal{M}$ we may appeal to Lemma 5.4.3 in place of Lemma 5.4.2. By doing so, we conclude that $-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-\Lambda) \times [0, T]$. Again $R \geq \Lambda$ means that $\left\lfloor \frac{R}{\Lambda} \right\rfloor \geq 1$, and so we have established the Gauss curvature estimates required in (5.5.2) throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda\right) \times [0, T]$, giving the required improvement.

For the remainder of the proof we assume that $T > \varepsilon$. We proceed under the assumptions that $g(0)$ satisfies both the estimates specified in (5.5.1), and will only later make a single extra step to remove the time delay before we obtain the estimates in (5.5.2) when we have the initial equality $g(0) \equiv \mathcal{H}$. Our first goal is to establish that the flow $g(t)$ satisfies the barriers $(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H}$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2\right) \times [0, \mathcal{T}_{\max}]$, where $\mathcal{T}_{\max}$ is as specified in (5.5.3). To achieve this, we will inductively apply Lemma 5.4.1 followed by Lemma 5.4.2 to rescalings of $g(t)$.

To illustrate, note we have the required hypotheses to appeal to Lemma 5.4.1 and deduce, since $\min\{T, \varepsilon\} = \varepsilon$ now, that we have the barriers $(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H}$ throughout $B_{\mathcal{H}}(x, R-\Lambda+2) \times [0, \varepsilon]$. These barriers allow us to apply Lemma 5.4.2 to the flow $g(t)$ to obtain that $-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-\Lambda) \times [\delta, \varepsilon]$. Since $\alpha \in (0, 1]$, these Gauss curvature estimates tell us that $\left| \mathrm{K}_{\frac{g(\varepsilon)}{1+2\varepsilon}} \right| \leq 2$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-\Lambda)$. Therefore the metric $\frac{g(\varepsilon)}{1+2\varepsilon}$ satisfies the same barriers and Gauss curvature bounds throughout $\mathbb{B}_{\mathcal{H}}(x, R-\Lambda)$ as those satisfied by $g(0)$ throughout $\mathbb{B}_{\mathcal{H}}(x, R)$. Hence it is natural to try to apply Lemma 5.4.1 to a rescaling of the flow $g(t)$ which takes $\frac{g(\varepsilon)}{1+2\varepsilon}$ as its initial metric.

The rescaled Ricci flow $\tilde{g}(s)$ given by $\tilde{g}(s) := \frac{g(\varepsilon+(1+2\varepsilon)s)}{1+2\varepsilon}$, defined on $\mathcal{M}$ for all $s \in \left[0, \frac{T-\varepsilon}{1+2\varepsilon}\right]$, satisfies that $\tilde{g}(0) = \frac{g(\varepsilon)}{1+2\varepsilon}$ as required. Thus it is to this flow that we aim to apply first Lemma 5.4.1, and then Lemma 5.4.2. Modulo checking that all of the required hypotheses are satisfied (which we will later do rigorously), the relationship between ε and $\frac{T-\varepsilon}{1+2\varepsilon}$ will determine whether this subsequent application of Lemmas 5.4.1 and 5.4.2 establishes control up until time T, or if the flow $\tilde{g}(s)$ exists beyond $s = \varepsilon$, which itself corresponds to having $T > \varepsilon + (1+2\varepsilon)\varepsilon$.

We also need to consider how the spatial region is changing. Each time we appeal to Lemma 5.4.1, followed by Lemma 5.4.2, we require being able to move in to a spatial $\mathcal{H}$ ball, centred at x, of radius Λ less than the original radius. Therefore we can only make this application of Lemma 5.4.1, followed by Lemma 5.4.2, to the flow $\tilde{g}(s)$ if we have that $R - \Lambda \geq \Lambda$, i.e. if $R - 2\Lambda \geq 0$.

If both $T > \varepsilon + (1+2\varepsilon)\varepsilon$ and $R - 2\Lambda \geq 0$ are true, we could apply the lemmas as specified above to control the Ricci flow $\tilde{g}(s)$ up until $s = \varepsilon$. The aim would then be to repeat this procedure by considering a rescaling of $\tilde{g}(s)$ taking $\frac{\tilde{g}(\varepsilon)}{1+2\varepsilon}$ as its initial metric.

In order to implement this iterative process we introduce some notation. We define $q \in \mathbb{N}_0$ to be the value

$$q := \max\left\{ l \in \mathbb{N}_0 : \sum_{k=0}^{l} \varepsilon(1+2\varepsilon)^k \leq T \right\}, \tag{5.5.4}$$

which is possible since we are assuming $T > \varepsilon$. Let $N := \min\left\{q, \left\lfloor \frac{R}{\Lambda} \right\rfloor - 1\right\}$. We will later see that $N+1$ corresponds to the maximum number of times we may iteratively appeal first to Lemma 5.4.1, followed by Lemma 5.4.2, to establish the required barriers over a time interval of size ε, and the Gauss curvature control at the later time ε. For now, we observe that we necessarily have that $R - (N+1)\Lambda \geq 0$, hence $R - i\Lambda \geq 0$ for every $i \in \{1, \ldots, N+1\}$.

For notational convenience we set $g_0(t) := g(t)$ for $t \in [0, \varepsilon]$. and recall that we have established that $(1-b)\mathcal{H} \leq \frac{g_0(t)}{1+2t} \leq (1+b)\mathcal{H}$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R - \Lambda + 2) \times [0, \varepsilon]$ and that $\left|\mathrm{K}_{\frac{g_0(\varepsilon)}{1+2\varepsilon}}\right| \leq 2$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R - \Lambda)$.

For $i \in \{1, \ldots, N+1\}$ we define

$$\tau_i := \frac{T - \sum_{k=0}^{i-1} \varepsilon(1+2\varepsilon)^k}{(1+2\varepsilon)^i} \tag{5.5.5}$$

which will correspond to the (rescaled) remaining existence time for the flow $g(t)$ after having made i applications of Lemmas 5.4.1 and 5.4.2. Naturally this means that $\tau_i > \tau_{i+1}$ when both are defined, and further we claim that $\tau_i \geq \varepsilon$ for every $i \in \{1, \ldots, N\}$. To see this observe that $q \geq N$, and hence from (5.5.4) we know that $T \geq \sum_{k=0}^{q} \varepsilon(1+2\varepsilon)^k \geq \sum_{k=0}^{N} \varepsilon(1+2\varepsilon)^k$. Therefore, if $i \in \{1, \ldots, N\}$, we can compute, using (5.5.5), that

$$\tau_i := \frac{T - \sum_{k=0}^{i-1} \varepsilon(1+2\varepsilon)^k}{(1+2\varepsilon)^i} \geq \frac{\sum_{k=0}^{N} \varepsilon(1+2\varepsilon)^k - \sum_{k=0}^{N-1} \varepsilon(1+2\varepsilon)^k}{(1+2\varepsilon)^N} = \varepsilon$$

as required. For $i \in \{1, \ldots, N\}$ we inductively define

$$g_i(t) := \frac{g_{i-1}(\varepsilon + (1+2\varepsilon)t)}{1+2\varepsilon} \tag{5.5.6}$$

which is a smooth Ricci flow defined on $\mathcal{M}$ for all $t \in [0, \tau_i]$. Previously, we have seen that $g_1(t)$ is defined on $\mathcal{M}$ for all $t \in [0, \tau_1]$. Then observe, for $i \in \{1, \ldots, N-1\}$, that if $g_i(t)$ is defined on $\mathcal{M}$ for all $[0, \tau_i]$, then from (5.5.6) we see that $g_{i+1}(t)$ is defined on $\mathcal{M}$ for all $t \in [0, t_*]$ where t_* satisfies that $\varepsilon + (1+2\varepsilon)t_* = \tau_i$. Hence $t_* = \frac{\tau_i - \varepsilon}{1+2\varepsilon} = \tau_{i+1}$ as required.

Recall that by assumption we have that for any $z \in \mathbb{B}_{\mathcal{H}}(x, R - i\Lambda)$ and all $t \in [0, T]$ that

$\mathbb{B}_{g(t)}\left(z,(1+2\varepsilon)^{\frac{i}{2}}\right)\subset\subset\mathcal{M}$. This tells us that for any $z\in\mathbb{B}_{\mathcal{H}}(x,R-i\Lambda)$ and all $t\in[0,\tau_i]$ we have

$$\mathbb{B}_{g_i(t)}(z,1)=\mathbb{B}_{g\left(\sum_{k=0}^{i-1}\varepsilon(1+2\varepsilon)^k+(1+2\varepsilon)^i t\right)}\left(z,(1+2\varepsilon)^{\frac{i}{2}}\right)\subset\subset\mathcal{M}. \tag{5.5.7}$$

Recall that we have established both that $(1-b)\mathcal{H}\leq\frac{g_0(\varepsilon)}{1+2\varepsilon}\leq(1+b)\mathcal{H}$ and $\left|\mathrm{K}_{\frac{g_0(\varepsilon)}{1+2\varepsilon}}\right|\leq 2$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-\Lambda)$. In terms of $g_1(t)$, these give that $(1-b)\mathcal{H}\leq g_1(0)\leq(1+b)\mathcal{H}$ and $|\mathrm{K}_{g_1(0)}|\leq 2$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-\Lambda)$. These estimates, together with the compact inclusions in (5.5.7) (for $i=1$), provide the required hypotheses to apply Lemma 5.4.1 to the flow $g_1(t)$.

In fact, we may proceed inductively, with the following claim giving the inductive step.

Claim: [Inductive step] Suppose $i\in\{1,\ldots,N\}$ and we have both $(1-b)\mathcal{H}\leq g_i(0)\leq(1+b)\mathcal{H}$ and $|\mathrm{K}_{g_i(0)}|\leq 2$ throughout $\mathbb{B}_{\mathcal{H}}(x,R-i\Lambda)$. Then we have that

$$(1-b)\mathcal{H}\leq\frac{g_i(t)}{1+2t}\leq(1+b)\mathcal{H} \tag{5.5.8}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(i+1)\Lambda+2)\times[0,\varepsilon]$, and

$$-1-\alpha\leq\mathrm{K}_{\frac{g_i(t)}{1+2t}}\leq-1+\alpha \tag{5.5.9}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(i+1)\Lambda)\times[\delta,\varepsilon]$. Since $\alpha\in(0,1]$, a particular consequence of (5.5.9) is that we have $\left|\mathrm{K}_{\frac{g_i(\varepsilon)}{1+2\varepsilon}}\right|\leq 2$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(i+1)\Lambda)$.

Proof: The assumptions in the claim, combined with the compact inclusions of (5.5.7) for i, along with noting that $g_i(0)$ is conformal to $\mathcal{H}$, provide the required hypothesis to apply Lemma 5.4.1 to the flow $g_i(t)$. Since $\tau_i\geq\varepsilon$ we can deduce the barriers in (5.5.8) over $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(i+1)\Lambda+2)\times[0,\varepsilon]$ as required. The barriers in (5.5.8), along with noting that $0<\delta<\varepsilon\leq\tau_i$ and $R-(i+1)\Lambda+2\geq 2$, allow us to appeal to Lemma 5.4.2 to deduce the Gauss curvature estimates (5.5.9) throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(i+1)\Lambda)\times[\delta,\varepsilon]$ as claimed. ††

By appealing to the inductive step in the claim a total of N times, observing that the conclusions of the claim for $i\in\{1,\ldots,N-1\}$ provide the required hypothesis in order to appeal to the claim for $i+1$, we can deduce the barriers in (5.5.8) for every $i\in\{1,\ldots,N\}$, along with already having established such barriers for $i=0$. Recalling (5.5.6), we can compute that for $i\in\{1,\ldots,N\}$ and $s\in[0,\varepsilon]$ we have

$$\frac{g_i(s)}{1+2s}=\frac{g\left(\sum_{k=0}^{i-1}\varepsilon(1+2\varepsilon)^k+(1+2\varepsilon)^i s\right)}{1+2\left(\sum_{k=0}^{i-1}\varepsilon(1+2\varepsilon)^k+(1+2\varepsilon)^i s\right)} \tag{5.5.10}$$

where we have used that $1+2\varepsilon\sum_{k=0}^{i-1}(1+2\varepsilon)^k=(1+2\varepsilon)^i$. Thus (5.5.8) tells us that

$$(1-b)\mathcal{H}\leq\frac{g(t)}{1+2t}\leq(1+b)\mathcal{H} \tag{5.5.11}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(i+1)\Lambda+2)\times\left[\sum_{k=0}^{i-1}\varepsilon(1+2\varepsilon)^k,\sum_{k=0}^{i}\varepsilon(1+2\varepsilon)^k\right]$. Combining (5.5.11) for each $i\in\{1,\ldots,N\}$, and recalling that we already know that $(1-b)\mathcal{H}\leq\frac{g(t)}{1+2t}\leq(1+b)\mathcal{H}$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-\Lambda+2)\times[0,\varepsilon]$, yields that

$$(1+b)\mathcal{H}\leq\frac{g(t)}{1+2t}\leq(1+b)\mathcal{H} \tag{5.5.12}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(N+1)\Lambda+2)\times\left[0,\sum_{k=0}^{N}\varepsilon(1+2\varepsilon)^k\right]$.

We must now split into two cases depending on the value taken by N. If $N=\lfloor\frac{R}{\Lambda}\rfloor-1$ then we do not have sufficient spatial room left to appeal to the claim. In this case we can compute that

$$\sum_{k=0}^{N}\varepsilon(1+2\varepsilon)^k=\frac{1}{2}\left(\exp\left[(N+1)\log(1+2\varepsilon)\right]-1\right),$$

and since $N=\lfloor\frac{R}{\Lambda}\rfloor-1$ we see that this gives the form of $\mathcal{T}_{max}$ as claimed in (5.5.3). Hence we have established the barriers of (5.5.12) throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x,R-\lfloor\frac{R}{\Lambda}\rfloor\Lambda+2\right)\times[0,\mathcal{T}_{\max}]$.

If $N<\lfloor\frac{R}{\Lambda}\rfloor-1$ then we still have the spatial room required to appeal to the claim. However, in this case we necessarily have that $N=q$ and so $\tau_{N+1}<\varepsilon$, hence we can only establish control up to time τ_{N+1}. Indeed, consider the rescaled Ricci flow

$$g_{N+1}(t):=\frac{g_N(\varepsilon+(1+2\varepsilon)t)}{1+2\varepsilon} \tag{5.5.13}$$

defined on $\mathcal{M}$ for all $t\in[0,\tau_{N+1}]$, where $g_N(t)$ is as defined in (5.5.6) for $i=N$.

Since we were able to apply the inductive step, as stated in the previous claim, to the flow $g_N(t)$, we know that we have both $(1-b)\mathcal{H}\leq\frac{g_N(\varepsilon)}{1+2\varepsilon}\leq(1+b)\mathcal{H}$ and $\left|\mathrm{K}_{\frac{g_N(\varepsilon)}{1+2\varepsilon}}\right|\leq 2$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(N+1)\Lambda)$. Therefore, from (5.5.13) we see that these estimates tell us that we have both $(1-b)\mathcal{H}\leq g_{N+1}(0)\leq(1+b)\mathcal{H}$ and $\left|\mathrm{K}_{g_{N+1}(0)}\right|\leq 2$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(N+1)\Lambda)$. Hence the compact inclusions in (5.5.7) for $i=N+1$, and the fact that $g_{N+1}(0)$ is conformal to $\mathcal{H}$, combine with the above estimates to provide the required hypotheses to apply Lemma 5.4.1 to the flow $g_{N+1}(t)$. Doing so yields, recalling that $\tau_{N+1}<\varepsilon$, that

$$(1-b)\mathcal{H}\leq\frac{g_{N+1}(t)}{1+2t}\leq(1+b)\mathcal{H} \tag{5.5.14}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x,R-(N+2)\Lambda+2)\times[0,\tau_{N+1}]$. Repeating the computations in (5.5.10) and

(5.5.11) for $i = N+1$ we see that (5.5.14) yields that

$$(1-b)\mathcal{H} \le \frac{g(t)}{1+2t} \le (1+b)\mathcal{H} \tag{5.5.15}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}(x, R-(N+2)\Lambda+2) \times \left[\sum_{k=0}^{N} \varepsilon(1+2\varepsilon)^k, \sum_{k=0}^{N} \varepsilon(1+2\varepsilon)^k + (1+2\varepsilon)^{N+1}\tau_{N+1}\right]$. From (5.5.5) we can compute that

$$\sum_{k=0}^{N} \varepsilon(1+2\varepsilon)^k + (1+2\varepsilon)^{N+1}\tau_{N+1} = T,$$

and since $N < \left\lfloor \frac{R}{\Lambda} \right\rfloor - 1$ we must have that $R-(N+2)\Lambda+2 \ge R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2$. These observations allow us to combine (5.5.12) with (5.5.15) to deduce that $(1-b)\mathcal{H} \le \frac{g(t)}{1+2t} \le (1+b)\mathcal{H}$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2\right) \times [0, T]$. Since $\mathcal{T}_{\max} \le T$, we have these barriers for all times $t \in [0, \mathcal{T}_{\max}]$.

In either case we have established that

$$(1-b)\mathcal{H} \le \frac{g(t)}{1+2t} \le (1+b)\mathcal{H} \tag{5.5.16}$$

throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2\right) \times [0, \mathcal{T}_{\max}]$. We will now use these barriers and Lemma 5.4.2 to establish the Gauss curvature estimates required in (5.5.2) throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda\right) \times [\delta, \mathcal{T}_{\max}]$. Consider any $s \in [\delta, \mathcal{T}_{\max}]$ and define $\gamma_s := \frac{s-\delta}{1+2\delta} \in [0, s)$. Then we may consider the Ricci flow $g_s(t) := \frac{g(\gamma_s + (1+2\gamma_s)t)}{1+2\gamma_s}$ on $\mathcal{M}$, defined for all times $t \in \left[0, \frac{\mathcal{T}_{\max}-\gamma_s}{1+2\gamma_s}\right]$, and with $g_s(0)$ conformal to $\mathcal{H}$. Observe that

$$\begin{aligned}
\frac{\mathcal{T}_{\max}-\gamma_s}{1+2\gamma_s} - \delta &= \frac{\mathcal{T}_{\max}-\gamma_s-\delta-2\gamma_s\delta}{1+2\gamma_s} \\
&= \frac{(1+2\delta)\mathcal{T}_{\max} - (s-\delta) - (1+2\delta)\delta - 2(s-\delta)\delta}{(1+2\gamma_s)(1+2\delta)} \\
&= \frac{1+2\delta}{1+2s}(\mathcal{T}_{\max} - s) \ge 0
\end{aligned}$$

where we have used that $1+2\gamma_s = \frac{1+2s}{1+2\delta}$. Hence the flow $g_s(t)$ is defined, at least, up to time δ, and we restrict to only considering $g_s(t)$ for times $t \in [0, \delta]$. A computation yields that for $t \in [0, \delta]$

$$\frac{g_s(t)}{1+2t} = \frac{g(\gamma_s + (1+2\gamma_s)t)}{(1+2t)(1+2\gamma_s)} = \frac{g(\gamma_s + (1+2\gamma_s)t)}{1+2(\gamma_s + (1+2\gamma_s)t)} \tag{5.5.17}$$

where $\gamma_s + (1+2\gamma_s)t \le \gamma_s + (1+2\gamma_s)\delta = s \le \mathcal{T}_{\max}$. Therefore (5.5.16) tells us that $(1-b)\mathcal{H} \le \frac{g_s(t)}{1+2t} \le (1+b)\mathcal{H}$ throughout $\mathbb{B}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2\right) \times [0, \delta]$. Further, $\mathbb{B}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2\right) \subset \mathbb{B}_{\mathcal{H}}(x, R) \subset\subset \mathcal{M}$ by assumption. Clearly $R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2 \ge 2$ and hence, recalling how b was specified at the start of the proof, we may apply Lemma 5.4.2 to the flow $g_s(t)$ to obtain that

$-1-\alpha \leq \mathrm{K}_{\frac{g_s(\delta)}{1+2\delta}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda\right)$. Using (5.5.17) for $t = \delta$ yields that $\frac{g_s(\delta)}{1+2\delta} = \frac{g(s)}{1+2s}$, and so the Gauss curvature control for $\frac{g_s(\delta)}{1+2\delta}$ tells us that $-1-\alpha \leq \mathrm{K}_{\frac{g(s)}{1+2s}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda\right)$. Repeating for all $s \in [\delta, \mathcal{T}_{\max}]$ allows us to conclude that $-1-\alpha \leq \mathrm{K}_{\frac{g(s)}{1+2s}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda\right) \times [\delta, \mathcal{T}_{\max}]$, as required in(5.5.2).

If we are only assuming both the estimates in (5.5.1) for $g(0)$ throughout $\mathbb{B}_{\mathcal{H}}(x,R)$ we stop here and are done. If instead we are assuming $g(0) \equiv \mathcal{H}$ throughout $\mathcal{M}$, we make a final additional step to avoid any time delay before obtaining the Gauss curvature control claimed in (5.5.2). Indeed, we have that $(1-b)\mathcal{H} \leq \frac{g(t)}{1+2t} \leq (1+b)\mathcal{H}$ throughout $\mathbb{B}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2\right) \times [0,\varepsilon]$, and additionally we have $g(0) \equiv \mathcal{H}$ throughout $\mathcal{M}$ by assumption. Recalling how b was specified at the start of the proof, and noting that $R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda + 2 \geq 2$, we may appeal to Lemma 5.4.3 to conclude that $-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}} \leq -1+\alpha$ throughout $\overline{\mathbb{B}}_{\mathcal{H}}\left(x, R - \left\lfloor \frac{R}{\Lambda} \right\rfloor \Lambda\right) \times [0,\varepsilon]$. Combined with our previous Gauss curvature estimates, we obtain the Gauss curvature estimates in (5.5.2) for all times $t \in [0, \mathcal{T}_{\max}]$, i.e. we have removed the time delay as required. This completes the proof of Theorem 5.5.1. ∎

Proof of Theorem 5.1.6. Retrieve the universal constant $\varepsilon > 0$ arising in Theorem 5.5.1. Let $\alpha \in (0,1]$ and $\delta \in (0,\varepsilon)$. Take $\Lambda = \Lambda(\alpha,\delta) > 0$ and $b = b(\alpha,\delta) > 0$ to be the respective constants arising in Theorem 5.5.1. We may now define

$$c = c(\alpha,\delta) := \frac{1}{4\Lambda}\log(1+2\varepsilon) > 0 \tag{5.5.18}$$

and

$$\mathcal{R} = \mathcal{R}(\alpha,\delta) := \max\left\{\left(1 + \frac{2}{\log(1+2\varepsilon)}\right)\Lambda, 4\Lambda\frac{\log(2\sqrt{1+2\varepsilon})}{\log(1+2\varepsilon)}\right\} \geq \Lambda > 0. \tag{5.5.19}$$

Now assume that $R \geq \mathcal{R}$ and $(\mathcal{M},\mathcal{H})$ is a smooth surface which satisfies that, for some $x \in \mathcal{M}$, the ball $\mathbb{B}_{\mathcal{H}}(x,R) \subset\subset \mathcal{M}$ and $(\mathbb{B}_{\mathcal{H}}(x,R),\mathcal{H})$ is isometric to a hyperbolic disc of radius R. Suppose $g(t)$ is a complete smooth Ricci flow on $\mathcal{M}$, defined for all $t \in [0,T]$ for some $T > 0$, with $g(0)$ conformal to $\mathcal{H}$, and satisfying that $(1-b)\mathcal{H} \leq g(0) \leq (1+b)\mathcal{H}$ and $|\mathrm{K}_{g(0)}| \leq 2$ throughout $\mathbb{B}_{\mathcal{H}}(x,R)$. From (5.5.19) we have that $R \geq \mathcal{R} \geq \Lambda$. Therefore we may appeal to Theorem 5.5.1 to obtain, recalling (5.5.2) and (5.5.3), that at the point $x \in \mathcal{M}$ we have $-1-\alpha \leq \mathrm{K}_{\frac{g(t)}{1+2t}}(x) \leq -1+\alpha$ for all times $\delta \leq t \leq \tilde{\mathcal{T}}_{\max}$ where

$$\tilde{\mathcal{T}}_{\max} := \min\left\{T, \frac{1}{2}\left(\exp\left[\left\lfloor \frac{R}{\Lambda} \right\rfloor \log(1+2\varepsilon)\right] - 1\right)\right\}. \tag{5.5.20}$$

Observe that (5.5.19) gives that $R \geq \mathcal{R} \geq \left(1 + \frac{2}{\log(1+2\varepsilon)}\right)\Lambda$. Therefore $\left(\frac{R}{\Lambda} - 1\right)\log(1+2\varepsilon) \geq 2$

and thus

$$\exp\left[\left\lfloor\frac{R}{\Lambda}\right\rfloor\log(1+2\varepsilon)\right]-1 \geq \exp\left[\left(\frac{R}{\Lambda}-1\right)\log(1+2\varepsilon)\right]-1 \geq \exp\left[\frac{1}{2}\left(\frac{R}{\Lambda}-1\right)\log(1+2\varepsilon)\right] \tag{5.5.21}$$

since $e^x - 1 \geq e^{\frac{x}{2}}$ for $x \geq 2$.

For $x, y > 0$ we have $\frac{1}{x}e^y \geq e^{\frac{y}{2}}$ provided $y \geq 2\log(x)$. Observe that $R \geq \mathcal{R} \geq 4\Lambda\frac{\log(2\sqrt{1+2\varepsilon})}{\log(1+2\varepsilon)}$ from (5.5.19), and so $\frac{R}{2\Lambda}\log(1+2\varepsilon) \geq 2\log(2\sqrt{1+2\varepsilon})$. Thus, using the above inequality with $x := 2\sqrt{1+2\varepsilon}$ and $y := \frac{R}{2\Lambda}\log(1+2\varepsilon)$, we deduce that

$$\frac{1}{2\sqrt{1+2\varepsilon}}\exp\left[\frac{R}{2\Lambda}\log(1+2\varepsilon)\right] \geq \exp\left[\frac{R}{4\Lambda}\log(1+2\varepsilon)\right] = e^{cR}, \tag{5.5.22}$$

recalling the definition of $c > 0$ in (5.5.18). Finally we can compute that

$$\begin{aligned}
\tilde{\mathcal{T}}_{\max} &\overset{(5.5.20)}{=} \min\left\{T, \frac{\exp\left[\left\lfloor\frac{R}{\Lambda}\right\rfloor\log(1+2\varepsilon)\right]-1}{2}\right\} \\
&\overset{(5.5.21)}{\geq} \min\left\{T, \frac{1}{2}\exp\left[\frac{1}{2}\left(\frac{R}{\Lambda}-1\right)\log(1+2\varepsilon)\right]\right\} \\
&= \min\left\{T, \frac{1}{2\sqrt{1+2\varepsilon}}\exp\left[\frac{R}{2\Lambda}\log(1+2\varepsilon)\right]\right\} \overset{(5.5.22)}{\geq} \min\left\{T, e^{cR}\right\} =: \mathcal{T}_{\max}
\end{aligned}$$

as claimed in (5.1.3) in Theorem 5.1.6. ∎

Proof of Theorem 5.1.1. Retrieve the universal constant $\varepsilon > 0$ arising in Theorem 5.5.1. Let $\alpha \in (0,1]$ be given and take $\delta := \frac{\varepsilon}{2} \in (0,\varepsilon)$. For this choice of δ we can retrieve constants $\Lambda = \Lambda(\alpha) > 0$ and $b = b(\alpha) > 0$ from Theorem 5.5.1. Using these constants, we can define $c > 0$ and $\mathcal{R} > 0$ exactly as they are defined in (5.5.18) and (5.5.19) respectively, now both depending only on α as required. Repeat the proof of Theorem 5.1.6, observing that, in the notation of Theorem 5.5.1, we now assume that $g(0) \equiv \mathcal{H}$ throughout $\mathcal{M}$, and so we may now use the version of Theorem 5.5.1 that avoids any time delay before achieving the desired Gauss curvature control. Proceeding verbatim as in the proof of Theorem 5.1.6 above establishes that we have the Gauss curvature estimates claimed in (5.1.1) at $x \in \mathcal{M}$ for the time required in (5.1.1) in Theorem 5.1.1. ∎

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